

Solve three of the following problems.

23. Let A be the disk algebra: $A = \overline{\text{span}}\{1, z, z^2, \dots\} \subseteq C(\overline{\mathbb{D}})$. Let $f_1, \dots, f_n \in A$ be such that $\sum_{i=1}^n |f_i(z)| \neq 0$ for all $z \in \overline{\mathbb{D}}$. Prove that there are $g_1, \dots, g_n \in A$ such that $\sum_{i=1}^n f_i(z)g_i(z) = 1$ for all $z \in \overline{\mathbb{D}}$. (Hint: consider the ideal in A generated by f_1, \dots, f_n .)
24. (a) Let A be a unital Banach algebra, let $x \in A$, and let Ω_∞ be the unbounded component of $\mathbb{C} \setminus \sigma_A(x)$. Prove that for every $\lambda \in \Omega_\infty$ there is a sequence of polynomials p_n such that $p_n(x) \rightarrow (x - \lambda)^{-1}$ in norm.
- (b) Let X be a compact subset of \mathbb{C} , let p, q be polynomials, and assume that the roots of q lie in the unbounded component of $\mathbb{C} \setminus X$. Prove that there is a sequence of polynomials p_n such that $p_n \rightarrow p/q$ uniformly on X .
25. (a) Let V be a complex vector space, and let $[\cdot, \cdot] : V \times V \rightarrow \mathbb{C}$ be a sesquilinear form. Prove the *polarization identity*:

$$[x, y] = \frac{1}{4} \sum_{k=0}^3 [x + i^k y, x + i^k y].$$

- (b) Let H be a Hilbert space, and $T \in B(H)$. The *quadratic form* of T is the function $q_T : H \rightarrow \mathbb{C}$ defined by $q_T(\xi) = \langle T\xi, \xi \rangle$. The *numerical radius* of T is defined by $w(T) = \sup\{|q_T(\xi)| : \|\xi\| = 1\}$. Prove that $w(T) \leq \|T\| \leq 2w(T)$.
- (c) Let $S, T \in B(H)$. Prove that $q_S = q_T$ if and only if $S = T$.
26. Let $T \in B(H)$, and suppose that T commutes with every operator in $B(H)$. Prove that T is a scalar multiple of the identity operator.