MAT 578 **HOMEWORK 5** **Due: 9/27/07**

Solve three of the following problems.

Let A be a unital commutative Banach algebra.

19. Prove that the Gelfand map Γ_A is an isometry if and only if $\|x^2\| = \|x\|^2$ for all $x \in A$.

20. Define the *radical* of A to be the set of quasinilpotent elements of A: $\operatorname{rad} A = \big\{x \in A : \sigma(x) = \{0\}\big\}$. A is called *semisimple* if $\operatorname{rad} A = \{0\}$.

    Prove that $\operatorname{rad} A$ is a closed ideal in A, and that $A/\operatorname{rad} A$ is semisimple.

21. Let A and B be commutative unital Banach algebras, and let $\phi : A \to B$ be a homomorphism (not assumed continuous), such that $\phi(1_A) = 1_B$.
    (a) Define $\widehat{\theta} : \operatorname{sp}(B) \to \operatorname{sp}(A)$ by $\widehat{\theta}(\omega) = \omega \circ \theta$. Prove that $\widehat{\theta}$ is continuous.
    (b) Suppose that B is semisimple. Prove that θ is continuous. (Use the closed graph theorem.)

22. Find (with proof) a condition analogous to the one in problem 19, that is equivalent to the Gelfand map Γ_A being bounded below.