MAT 578 **HOMEWORK 2** **Due: 9/6/07**

Solve three of the following problems.

5. (a) A *topological group* is a group that is also a Hausdorff topological space such that the group operations (composition, inverse) are continuous. Let G be a topological group and let H be a subgroup which is an open set. Prove that H is a closed set.
   (b) Let A be a unital Banach algebra in which $\|1_A\| = 1$. Let G_0 be the subgroup of $G(A)$ generated by elements of the form $1 - x$ and $(1 - x)^{-1}$ with $x \in A$ and $\|x\| < 1$. Prove that G_0 equals the connected component of $G(A)$ containing 1_A.
   (c) With notation as in part (b), prove that G_0 is a normal subgroup of $G(A)$.

6. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of complex numbers. Prove that there is a bounded operator A on ℓ^2 such that $Ae_n = a_n e_{n+1}$, (where $e_n = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ is the nth standard basis vector in ℓ^2). Compute the norm of A in terms of (a_n). (The operator A is called a *unilateral weighted shift.*)

7. A *unitary* operator on a Hilbert space H is an operator $U \in B(H)$ such that U is an isometry and is invertible (and hence U^{-1} is also a unitary operator). Let A be a unilateral weighted shift (as in problem 6).
   (a) Prove that for every complex number λ with $|\lambda| = 1$, there is a unitary operator $U_\lambda \in B(\ell^2)$ such that $U_\lambda A U_\lambda^{-1} = \lambda A$.
   (b) Prove that the spectrum of A equals the union of (possibly degenerate) circles with center at 0.

8. Let A be a unilateral weighted shift corresponding to a sequence (a_n) as in problem 6. Suppose that $\lim_{n\to\infty} a_n = 0$. Prove that A is quasinilpotent.

9. Prove that the Volterra operator of problem 4 is quasinilpotent.