MAT 578 **HOMEWORK 13** **Due: 11/29/07**

Solve three problems from among the following, and from among any unworked problems on assignments 9, 10, 11 and 12.

55. Let T be a (densely defined) symmetric operator in H. Prove that $U_{\overline{T}} = \overline{U_T}$ (where $S \mapsto U_S$ is the Cayley transform).

56. Let T be a self-adjoint operator in H (with dense domain $D(T)$). Prove that for $x \in D(T)$,
$$iTx = \lim_{t \to 0} \frac{1}{t}(e^{itT}x - x).$$

57. Let T be the operator $-\frac{d^2}{dt^2}$ on $L^2(\mathbb{R})$, with domain $D(T)$ equal to the Schwartz space $\mathcal{S}$ (of infinitely differentiable functions which, together with all their derivatives, tend to zero at infinity more rapidly than any inverse power of x). Prove that T is essentially self-adjoint. (Hints: To show that $R(T \pm i)$ is dense, let $\psi \in C_c^\infty(\mathbb{R})$, and use variation of parameters to solve the differential equation $-\phi'' \pm i\phi = \psi$ for a function $\phi \in \mathcal{S}$.)

58. Let T be as in the previous problem, and let T_0 be the operator with domain $D(T_0)$ equal to the space $C_c^\infty(\mathbb{R})$ of all infinitely differentiable functions with compact support, and given by the same formula as T. Prove that T_0 is essentially self-adjoint. (Hints: Prove that $T \subseteq \overline{T_0}$ as follows. Let $\theta \in C_c^\infty(\mathbb{R})$ be a function such that $\theta = 1$ on $[-1, 1]$, θ is supported in $[-2, 2]$, and $0 \leq \theta \leq 1$ on $\mathbb{R}$. Let $\theta_n(x) = \theta(x/n)$. For $\psi \in \mathcal{S}$, prove that $\theta_n\psi \to \psi$ and $(\theta_n\psi)'' \to \psi''$ in $L^2(\mathbb{R})$.)