MAT 578 **HOMEWORK 12** **Due: 11/15/07**

Solve three problems from among the following, and from among any unworked problems on assignments 9, 10, and 11.

50. Let $c \in \mathbb{C}$ with $|c| = 1$. Define the operator T_c in the Hilbert space $L^2[0,1]$ by
$$D(T_c) = \{f \in AC[0,1] : f' \in L^2,\ f(1) = cf(0)\}$$
$$T_c f = \frac{1}{i} f'.$$
Prove that T_c is self-adjoint. (Hints: if $c \neq 1$, show that for every $h \in L^2$ there is $H \in AC[0,1]$ with $H' = h$ and $H(1) = cH(0)$. If $c = 1$, consider the constant function 1 in $D(T_1)$.)

51. (a) Let S and T be operators in H with $T \subseteq S$ and S symmetric. Suppose that $R(S+i) = R(T+i)$. Prove that $S = T$.
(b) Let A be a symmetric operator in H such that $R(A+i) = H$ and $R(A-i) \neq H$. Prove that A has no self-adjoint extension.

52. Let T be a symmetric operator in H. Let $D \subseteq D(T)$ be a linear manifold that is dense in H. Suppose that $T|D$ is essentially self-adjoint.
(a) Prove that T is essentially self-adjoint.
(b) Prove that T and $T|D$ have the same closure.

53. (a) Let T be an operator in H such that $G(T)$ is dense in $H \times H$. What can you say about T^*?
(b) Prove that such operators exist.

54. Define an operator in ℓ^2 by $D(T) = \operatorname{span}\{e_n : n \in \mathbb{N}\}$ and $Te_n = e_1$ for all n. Prove that T is not closable.