MAT 578 **HOMEWORK 11** **Due: 11/8/07**

Solve three problems from among the following, and from among any unworked problems on assignments 9 and 10.

46. For $n \geq 2$ let J_n be the $n \times n$ matrix having 1's on the superdiagonal and 0's elsewhere (so that J_n is nilpotent of order n).
   (a) Find $\|J_n^k\|$ for all $n \geq 2$ and $k \geq 0$.
   (b) Let $T = \bigoplus_{n \geq 2} J_n$. Prove that $r(T) = 1$. Use this to prove that $\sigma(T) \neq \overline{\bigcup_{n \geq 2} \sigma(J_n)}$.
   (c) Prove that $\sigma(T) = \overline{\mathbb{D}}$. (Hint: show that the norm of an $n \times n$ matrix is greater than or equal to the 2-norm of any of its rows and columns.)

47. (This problem is devoted to the proof of Herglotz' Theorem.) Recall that if μ is a complex Borel measure on $\mathbb{T}$, the Fourier-Stieltjes transform of μ is the function $\widehat{\mu} : \mathbb{Z} \to \mathbb{C}$ given by $\widehat{\mu}(n) = \int_{\mathbb{T}} z^{-n}\, d\mu(z)$.

   A function $f : \mathbb{Z} \to \mathbb{C}$ is called a function *of positive type* if for every complex sequence $(z_n)_{n \in \mathbb{Z}}$ that is finitely non-zero, we have
$$\sum_{n,k} f(n-k) z_k \overline{z_n} \geq 0.$$
   (a) Prove that if μ is a finite positive measure then $\widehat{\mu}$ is a function of positive type.
   (b) Let $f : \mathbb{Z} \to \mathbb{C}$ be a function of positive type. Prove that there is a finite positive measure μ on $\mathbb{T}$ such that $f = \widehat{\mu}$.

   (Outline for (b): Let H_0 be the vector space of all finitely non-zero complex sequences. Define a sesquilinear from on H_0 by $[\xi, \eta] = \sum_{n,k} f(n-k)\xi_k \overline{\eta_n}$. Prove that $[\cdot,\cdot]$ is positive semi-definite. Let $N = \{\xi \in H_0 : [\xi,\xi] = 0\}$. Then $[\cdot,\cdot]$ descends to give an inner product $\langle \cdot, \cdot \rangle$ on H_0/N via $\langle \xi + N, \eta + n \rangle = [\xi, \eta]$. Let H be the completion of H_0/N for this inner product. Prove that the shift on H_0 induces a unitary operator U on H, and that $e_0 + N$ is a cyclic vector for U. Use the spectral measure of U, and the cyclic vector, to define a measure on $\mathbb{T}$, and prove that it has f as its Fourier-Stieltjes transform.)

48. Let H be an infinite dimensional Hilbert space.
   (a) Prove that the group of unitary operators on H is connected (in the norm topology).
   (b) Prove that the set of invertible operators on H is connected.
   (c) Prove that the set of Fredholm operators on H having index 0 is connected.

49. Let $T \in B(H)$. Prove that T has closed range if and only if 0 is an isolated point (or not present) in $\sigma(T^*T)$.