

Solve three of the following problems.

1. (a) Let R be a unital ring, and let $x, y \in R$. Prove that $1 - xy$ is invertible if and only if $1 - yx$ is invertible. (Hint: If $1 - xy$ is invertible, pretend that the “formal” infinite geometric series for $(1 - xy)^{-1}$ exists, and use it to guess a formula for $(1 - yx)^{-1}$.)
(b) Let A be a unital algebra, and let $x, y \in A$. Prove that $\sigma(xy) \setminus \{0\} = \sigma(yx) \setminus \{0\}$.
2. The *rank* of a linear map is defined to be the dimension of its range. A linear map has *finite rank* if its rank is finite. Let X and Y be Banach spaces, and let $T \in B(X, Y)$. Suppose that there are finite rank operators $T_n \in B(X, Y)$ such that $\|T - T_n\| \rightarrow 0$. Prove that T is a compact operator.
3. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence in \mathbb{C} . Define an operator A on ℓ^2 by $(A\xi)_n = a_n\xi_n$.
(a) Prove that T is bounded.
(b) Prove that T is a compact operator if and only if $a_n \rightarrow 0$.
4. Define the *Volterra* operator on $L^2[0, 1]$ by

$$(Vf)(x) = \int_0^x f(t) dt.$$

Prove that V is a compact operator. (Hint: use the Arzela-Ascoli theorem.)