

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions. **Reread** your proofs **before** copying them out to turn in; I really do mean that you should write (at least) one draft of each solution.

**29.** Let  $X$  be a metric space, and let  $f, f_n : X \rightarrow \mathbf{R}^k$  be functions. Suppose that  $f_n$  is continuous for each  $n$ ,  $K \subseteq X$  is compact, and  $f_n \rightarrow f$  uniformly on  $K$ .

(i) Prove that if  $U \subseteq \mathbf{R}^k$  is open and  $f(K) \subseteq U$ , then there exists  $N$  such that  $f_n(K) \subseteq U$  for all  $n \geq N$ .

(ii) Prove that  $f(K) \cup \bigcup_{n=1}^{\infty} f_n(K)$  is compact.

**30.** Is the sequence  $(f_n)_{n=1}^{\infty}$  (defined on  $[0, 1]$ ) uniformly convergent? Prove your answer.

$$(i) f_n(x) = \frac{x}{1 + nx^2}$$

$$(ii) f_n(x) = \frac{nx}{1 + nx^2}$$

$$(iii) f_n(x) = \frac{nx}{1 + n^2x^2}$$

**31.** In each part, discuss (with proof) the existence and continuity of  $f$  and  $f'$  at  $x = 0$ .

(i)  $f(x) = x^2 \sin(1/x)$  for  $x \neq 0$ ,  $f(0) = 0$ .

(ii)  $f(x) = x^{7/3} \sin(1/x)$  for  $x \neq 0$ ,  $f(0) = 0$ .

**32.** Let  $f : [0, \infty) \rightarrow \mathbf{R}$  be differentiable, and assume that  $\lim_{x \rightarrow \infty} f'(x) = 2$ . Prove that  $f$  is uniformly continuous on  $[0, \infty)$ .