MAT 472 — Spielberg **HOMEWORK 7** **Due: 10/19/06**

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions. **Reread** your proofs **before** copying them out to turn in; I really do mean that you should write (at least) one draft of each solution.

25. Let $f : X \to Y$ be a continuous function between metric spaces X and Y.
(i) Prove that for any set $E \subseteq X$, $f(\overline{E}) \subseteq \overline{f(E)}$.
(ii) Prove by example that equality in part (i) need not hold.
(iii) Suppose that $X = \mathbf{R}^n$ and that E is a bounded set. Prove that equality in (i) holds.

26. Let (X, d) be a metric space, let $f : X \to \mathbf{R}$, and let $a \in X$. Define the *oscillation* of f at a by

$$\operatorname{osc}(f, a) = \inf_{r>0}\Big(\sup\big\{|f(x) - f(y)| \;\big|\; x,\, y \in B_r(a)\big\}\Big).$$

(i) Prove that f is continuous at a if and only if $\operatorname{osc}(f, a) = 0$.

(ii) Prove that for any $c > 0$, the set $\big\{x \in X : \operatorname{osc}(f, x) \geq c\big\}$ is a closed subset of X.

27. Let X and X' be metric spaces, and let $f : X \to X'$ be a function. Consider two properties that f might have:

(A) f is uniformly continuous.

(B) For every Cauchy sequence (x_n) in X, the sequence $\big(f(x_n)\big)$ in X' is Cauchy.

(i) Prove that (A) implies (B).
(ii) Does (B) imply (A)? Prove your answer.

28. (i) Let X, X' be metric spaces and let $f : X \to X'$ be a function. Prove that each of the following two statements implies the other:

(a) f is uniformly continuous.
(b) Whenever $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ are two sequences in X with $\lim_{i\to\infty} d(x_i, y_i) = 0$, then $\lim_{i\to\infty} d'\big(f(x_i), f(y_i)\big) = 0$.

(ii) Suppose that $f : X \to \mathbf{R}$ is continuous and has the property that for every $\epsilon > 0$, the set $\big\{x \in X : \big|f(x)\big| \geq \epsilon\big\}$ is compact. Prove that f is uniformly continuous.