MAT 472 — Spielberg **HOMEWORK 5** **Due: 9/28/06**

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions. **Reread** your proofs **before** copying them out to turn in; I really do mean that you should write (at least) one draft of each solution.

17. Let (X, d) be a metric space. Prove that X is sequentially compact if and only if every infinite subset of X has a limit point.

18. Let (X, d) be a complete metric space. For nonempty subsets A, $B \subseteq X$ define the *distance* between A and B by

$$\operatorname{dist}(A, B) = \inf\{d(x, y) \mid x \in A,\ y \in B\}.$$

(i) Prove that if A is compact, B is closed, and $A \cap B = \emptyset$, then $\operatorname{dist}(A, B) > 0$.
(ii) Prove that the conclusion of part (i) may fail if A and B are closed but neither is compact.

19. Let (X, d) be a compact metric space, and let $\mathcal{U}$ be an open cover of X. Prove that there exists a postive number ϵ such that every closed ball in X of radius ϵ is contained in a set from $\mathcal{U}$.

20. Let (X, d) be a metric space, and let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ be a decreasing sequence of nonempty compact subsets of X. Let $K = \cap_i K_i$. Suppose that $c > 0$ is such that $\operatorname{diam}(K_i) \geq c$ for all i. Prove that $\operatorname{diam}(K) \geq c$.