MAT 472 — Spielberg **HOMEWORK 4** **Due: 9/21/06**

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions. **Reread** your proofs **before** copying them out to turn in; I really do mean that you should write (at least) one draft of each solution.

13. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers that are bounded. Prove that

$$\limsup_{n\to\infty} (a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

and that equality holds if (a_n) or (b_n) is convergent.

14. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Let

$$A = \{x \in \mathbf{R} : a_n < x \text{ for infinitely many } n\}.$$

Prove that $\inf A = \liminf_{n\to\infty} a_n$.

15. Let (X, d) be a metric space. For a nonempty subset $E \subseteq X$ define the *diameter* of E by

$$\text{diam}\,(E) = \sup\{d(x, y) : x, y \in E\}$$

(where the diameter of an unbounded set is defined to be $+\infty$).

(i) Prove that $\text{diam}\,(E) < \infty$ if and only if $\text{diam}\,(\overline{E}) < \infty$, for any nonempty $E \subseteq X$.

(ii) Prove that $\text{diam}\,(E) = \text{diam}\,(\overline{E})$ for any nonempty $E \subseteq X$.

16. Let (X, d) be a metric space. Prove that X is complete if and only if the following condition holds: for every decreasing sequence $F_1 \supseteq F_2 \supseteq \cdots$ of nonempty closed subsets of X with $\lim_{n\to\infty} \text{diam}\,(F_n) = 0$, there exists an element $a \in X$ such that

$$\bigcap_{n=1}^{\infty} F_n = \{a\}.$$