**MAT 472** **HOMEWORK 2** **Due: 9/7/06**

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions. **Reread** your proofs **before** copying them out to turn in; I really do mean that you should write (at least) one draft of each solution.

**5.** Let $a$, $b \in \mathbf{R}^n$ and let $r > 0$, $s > 0$ be given. Suppose that $B_r(a) = B_s(b)$. Prove that $a = b$ and $r = s$.

**6.** Recall the metric $d$ on the space $X = \prod_{n=1}^{\infty}\{0,\, 1\}$ defined in class: for $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ distinct elements of $X$, we put $k(x, y) = \min\{i \in \mathbf{N} \mid x_i \neq y_i\}$, and let

$$d(x, y) = \begin{cases} 1/k(x, y), & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

Let $x$, $y \in X$ and $r > 0$.

(i) Prove that if $d(x, y) \geq r$, then $B_r(x) \cap B_r(y) = \emptyset$.

(ii) Prove that if $d(x, y) < r$, then $B_r(x) = B_r(y)$.

(Hint: if $0 < r \leq 1$, then there exists $n \in \mathbf{N}$ with $1/(n + 1) < r \leq 1/n$.)

**7.** Let $\mathcal{B}$ be the collection of open balls in $\mathbf{R}^n$ having rational radius and having a center with all coordinates rational.

(i) Prove that $\mathcal{B}$ is a countable collection.

(ii) Prove that if $U$ is any open ball in $\mathbf{R}^n$, and if $x$ is any point of $U$, then there is a ball $V \in \mathcal{B}$ such that $x \in V$ and $V \subseteq U$.

**8.** In this problem, $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbf{R}^2$: $\|x\| = (x_1^2 + x_2^2)^{1/2}$. Define the *one-norm*, $\|\cdot\|_1$, on $\mathbf{R}^2$ by:

$$\|x\|_1 = |x_1| + |x_2|.$$

(i) Find positive constants $a$ and $b$ such that

$$a\|x\| \leq \|x\|_1 \leq b\|x\|,\ x \in \mathbf{R}^2.$$

(ii) Let $d$ be the usual (Euclidean) metric on $\mathbf{R}^2$, and let $d_1$ be the metric associated to $\|\cdot\|_1$. Prove that each open ball for the metric $d$ is an open set for the metric $d_1$.

(iii) Prove that each open ball for the metric $d_1$ is an open set for the metric $d$.

(Thus the two metrics define the same collection of open sets. It may be helpful for your intuition to sketch the open unit ball with center $(0, 0)$ for the metric $d_1$.)