MAT 472 — Spielberg **HOMEWORK 13** **Due: Tuesday, 12/5/06**

Choose two of the following three problems.

49. Let $f : [a, b] \to \mathbf{R}$ be continuous, and suppose that for every non-negative integer n we have $\int_a^b x^n f(x)\,dx = 0$. Prove that $f = 0$.

50. Let X be a compact metric space, and let $\mathcal{F} \subseteq C(X, \mathbf{R}^k)$.

(i) Suppose that $\mathcal{F}$ is equicontinuous on X. Prove that $\mathcal{F}$ is uniformly equicontinuous on X.

(ii) Suppose that $\mathcal{F}$ is equicontinuous and pointwise bounded on X. Prove that $\mathcal{F}$ is uniformly bounded on X.

51. Let X and X' be metric spaces. Recall that a function $f : X \to X'$ satisfies a *Lipschitz condition* (with Lipschitz constant M) if for all $x, y \in X$, $d'\big(f(x), f(y)\big) \le M\, d(x, y)$.

(i) Let $\mathcal{F}$ be a family of functions from X to $\mathbf{R}^k$ all satisfying a Lipschitz condition with a common Lipschitz constant. Prove that $\mathcal{F}$ is uniformly equicontinuous.

(ii) Let (a_n) be a bounded sequence in $\mathbf{R}$ and let (b_n) be an arbitrary sequence in $\mathbf{R}$. Let $f_n : \mathbf{R} \to \mathbf{R}$ be given by $f_n(x) = \sin(a_n x + b_n)$. Prove that (f_n) has a subsequence that converges uniformly on every bounded interval in $\mathbf{R}$. (You may use the fact that $|\sin s - \sin t| \le |s - t|$ for all $s, t \in \mathbf{R}$.)