MAT 472 — Spielberg **HOMEWORK 12** **Due: Tuesday, 11/28/06**

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions. **Reread** your proofs **before** copying them out to turn in; I really do mean that you should write (at least) one draft of each solution.

45. For a real number x let (x) denote the *fractional part* of x: $(x) = x - [x]$, where $[x]$ is the greatest integer function. Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}.$$

(i) Prove that f is continuous on $\mathbf{R} \setminus \mathbf{Q}$.
(ii) Prove that f is discontinuous on $\mathbf{Q}$.
(iii) Prove that f is Riemann integrable on $[a, b]$, for any compact interval $[a, b]$.

46. (i) Let (a_n) be a sequence of non-zero real numbers. Suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1. \quad \text{Let } c = \limsup_{n\to\infty} \; n \cdot \left(\left| \frac{a_{n+1}}{a_n} \right| - 1 \right).$$

Prove that $\sum a_n$ converges absolutely if $c < -1$.
(Hint: choose $c < \alpha < -1$, let $b_n = n^{\alpha}$, and consider the limit of $n(b_{n+1}/b_n - 1)$.)

(ii) Let p and q be real numbers with $0 < p < q - 1$. Prove the convergence of

$$\sum_{n=1}^{\infty} \frac{p(p+1)(p+2)\cdots(p+n-1)}{q(q+1)(q+2)\cdots(q+n-1)}.$$

47. Let (a_n) be a sequence of real numbers such that $a_n \neq 0$ for at least one integer n. Suppose that the power series $\sum a_n x^n$ has a positive radius of convergence, and let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for x in the interval of convergence. Prove that there exists $\delta > 0$ such that $f(x) \neq 0$ for all x with $0 < |x| < \delta$.

48. Let the power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ have positive (or infinite) radius of convergence R. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ for $|x-c| < R$. Prove that f is real analytic at every point of $(c-R, c+R)$. (Hint: For $b \in (c-R, c+R)$ let $0 < \delta < R - |b-c|$. For $x \in (b-\delta, b+\delta)$ expand $(x-c)^n = [(x-b) + (b-c)]^n$ using the binomial formula. Use the theorem from class on double series.)