

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions. **Reread** your proofs **before** copying them out to turn in; I really do mean that you should write (at least) one draft of each solution.

41. In all parts of this problem, prove that your answer is correct.

(i) Find a sequence of continuous functions $f_n : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow 0} f_n(x) \neq \lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} f_n(x),$$

and such that both iterated limits exist. Show that this is impossible if (f_n) converges pointwise to a function that is continuous at 0.

(ii) Find a sequence of continuous functions $f_n : [0, 1] \rightarrow \mathbf{R}$ such that f_n converges to 0 pointwise, the sequence of integrals $\int_0^1 f_n$ is bounded, and the sequence of integrals $\int_0^1 f_n$ does not converge.

(iii) Find a sequence of Riemann integrable functions on $[0, 1]$ that converge to a bounded function that is not Riemann integrable.

42. Let $a_n \geq 0$ and suppose that $\sum_{n=1}^{\infty} a_n$ converges.

(i) Prove that $\liminf_{n \rightarrow \infty} na_n = 0$.

(ii) Prove by means of an example that $\lim_{n \rightarrow \infty} na_n$ need not exist.

43. Let $a_n \geq 0$ and suppose that $\sum_{n=1}^{\infty} a_n$ diverges.

(i) Must $\sum_{n=1}^{\infty} \frac{a_n}{1 + n^2 a_n}$ diverge? converge? Prove your answer.

(ii) Must $\sum_{n=1}^{\infty} \frac{a_n}{1 + na_n}$ diverge? converge? Prove your answer.

44. (i) Let $\{k_n\}_{n=1}^{\infty}$ be a sequence of functions on \mathbf{R} with the following properties:

(a) k_n is continuous.

(b) $k_n \geq 0$.

(c) $\int_{-\infty}^{\infty} k_n = 1$ (where this means $\lim_{r \rightarrow \infty} \int_{-r}^r k_n = 1$).

(d) For all $r > 0$, $\lim_{n \rightarrow \infty} \int_{-r}^r k_n = 1$.

Let $f : [-1, 1] \rightarrow \mathbf{R}$ be Riemann integrable, and suppose that f is continuous at 0. Prove that

$$\lim_{n \rightarrow \infty} \int_{-1}^1 f \cdot k_n = f(0)$$

(ii) Given that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, prove that $k_n(x) = \sqrt{\frac{n}{\pi}} e^{-nx^2}$ has properties (a) – (d).