MAT 472 — Spielberg **HOMEWORK 10** **Due: 11/9/06**

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions. **Reread** your proofs **before** copying them out to turn in; I really do mean that you should write (at least) one draft of each solution.

37. Let $f : [0, \infty) \to \mathbf{R}$ be continuous, and suppose that $L \in \mathbf{R}$ such that

$$\lim_{t\to\infty} f(t) = L. \qquad \text{Prove that} \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t f(x)\,dx = L.$$

38. Let $f : [a, b] \to \mathbf{R}$ be continuous. Prove that there exists $c \in (a, b)$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

39. Let $f : [a, b] \to \mathbf{R}$ be continuous and non-negative. Prove that

$$\lim_{n\to\infty} \left(\int_a^b f(x)^n\,dx \right)^{\frac{1}{n}} = \|f\|_{sup}.$$

(Hint: First prove it for functions of the form $\lambda\chi_{[c,d]}$.)

40. (i) Prove that if $g : [a, b] \to \mathbf{R}$ is a step function, then

$$\lim_{n\to\infty} \int_a^b g(x)\,\sin(nx)\,dx = 0.$$

(ii) Let $f : [a, b] \to \mathbf{R}$ be a continuous function, and let $\epsilon > 0$. Prove that there is a step function $g : [a, b] \to \mathbf{R}$ such that $\big|f(x) - g(x)\big| < \epsilon$ for all $x \in [a, b]$. (Hint: use uniform continuity.)

(iii) Prove that if $f : [a, b] \to \mathbf{R}$ is a continuous function, then

$$\lim_{n\to\infty} \int_a^b f(x)\,\sin(nx)\,dx = 0.$$