MAT 570 **HW 9** **Due Tuesday, 11/8/05**

Solve three problems from among these and past unsolved problems.

41. (i) Find Hf when $f = \chi_{[0,1]} \in L^1(\mathbf{R})$.
(ii) Let $f \in L^1_{loc}(\mathbf{R})$, and assume that $Hf \in L^1(\mathbf{R})$. Prove that $f = 0$ a.e.

42. Let $f \in L^1_{loc}(\mathbf{R}^n)$, let $x \in \mathbf{R}^n$, and suppose that f is continuous at x. Prove that $x \in L_f$.

43. Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $g(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(i) Prove that f and g are differentiable on $[-1, 1]$.
(ii) Prove that $f \in BV[-1, 1]$.
(iii) Prove that $g \notin BV[-1, 1]$.

44. (Fubini) Let $f_n : [a, b] \to [0, \infty)$ be increasing functions, and suppose that $\sum_n f_n(b) < \infty$. Let $s(x) = \sum_n f_n(x)$. Prove that $s' = \sum_n f_n'$ a.e.
(Hints: let $s_n = \sum_1^n f_k$.
(i) Show that $s_n' \leq s'$. Use this to show that $f_n' \to 0$ a.e.
(ii) Choose n_k such that $\sum \big(s(b) - s_{n_k}(b)\big) < \infty$. Let $g_k = s - s_{n_k}$. Then (g_k) satisfies the same hypotheses as (f_n).)

45. Let F be the Cantor function on $[0, 1]$, and set $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. Let $[a_1, b_1]$, $[a_2, b_2]$, … be an enumeration of the closed subintervals of $[0, 1]$ having (distinct) rational endpoints. For each n, set

$$F_n(x) = F\left(\frac{x - a_n}{b_n - a_n}\right) \quad \text{and} \quad G(x) = \sum_{n=1}^{\infty} 2^{-n} F_n(x).$$

Prove that G is continuous and strictly increasing on $[0, 1]$, and that $G'(x) = 0$ almost everywhere. (Hint: use problem 44.)

(OVER)

46. Let I be an open interval in $\mathbf{R}$. A function $F : I \to \mathbf{R}$ is called *convex* if

$$F\big(tx + (1-t)y\big) \leq tF(x) + (1-t)F(y)$$

for all x, $y \in I$ and $0 \leq t \leq 1$. Prove the following.

(i) F is convex iff for all x, y, x', $y' \in I$ such that $x \leq x' < y'$ and $x < y \leq y'$,

$$\frac{F(y) - F(x)}{y - x} \leq \frac{F(y') - F(x')}{y' - x'}.$$

(ii) F is convex on I iff F is absolutely continuous on every compact subinterval of I and F' is increasing (on the set where it is defined).

(iii) If F is convex on I and $x_0 \in I$, there exists $c \in \mathbf{R}$ such that $F(x) - F(x_0) \geq c(x - x_0)$ for all $x \in I$.

(iv) (Jensen's inequality) If $(X, \mathcal{M}, \mu)$ is a probability space ($\mu(X) = 1$), if $g : X \to I$ is integrable, and if F is convex on I, then

$$F\Big(\int g\, d\mu\Big) \leq \int F \circ g\, d\mu.$$

(Hint: Let $x_0 = \int g\, d\mu$ and $t = g(x)$, and apply part (iii).)