MAT 570 **HW 8** **Due Tuesday, 10/25/05**

Solve three problems from among these and past unsolved problems.

36. (i) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, let $\mathcal{N}$ be a sub-σ-algebra of $\mathcal{M}$, and let $\nu = \mu\big|\mathcal{N}$. Prove that for $f \in L^1(\mu)$ there exists a ν-a.e. unique $\mathcal{N}$-measurable function g, integrable with respect to ν, such that for all $E \in \mathcal{N}$, $\int_E f\,d\mu = \int_E g\,d\nu$. (Hint: g is a Radon-Nikodym derivative.)

In the above situation we define Φ ($\equiv \Phi_{\mathcal{N},\mathcal{M}}$) : $L^1(\mu) \to L^1(\nu)$ by $\Phi(f) = g$. ($\Phi(f)$ is called the *conditional expectation* of f on $\mathcal{N}$.) Prove the following properties of the map Φ.

(ii) Φ is linear, positive ($f \geq 0$ implies that $\Phi(f) \geq 0$), and identical on functions that are already $\mathcal{N}$-measurable.

(iii) If $f \in L^1(\mu)$ and g is $\mathcal{N}$-measurable, then $fg \in L^1(\mu)$ if and only if $\Phi(f)g \in L^1(\nu)$. In this case, $\Phi(fg) = \Phi(f)g$.

(iv) Let $\mathcal{P} \subseteq \mathcal{N}$. Then $\Phi_{\mathcal{P},\mathcal{M}} = \Phi_{\mathcal{P},\mathcal{N}} \circ \Phi_{\mathcal{N},\mathcal{M}}$.

37. Let $(X, \mathcal{M})$ be a measurable space, and let $\phi : X \to X$ be bimeasurable (i.e. ϕ is bijective, and both ϕ and ϕ^{-1} are measurable). Let μ_1 and μ_2 be σ-finite measures on $(X, \mathcal{M})$ with $\mu_1 << \mu_2$.

(i) Prove that $\phi_*\mu_1 << \phi_*\mu_2$.

(ii) Prove that $\dfrac{d(\phi_*\mu_1)}{d(\phi_*\mu_2)} = \dfrac{d\mu_1}{d\mu_2} \circ \phi^{-1}$.

38. Let ν be a complex measure on $(X, \mathcal{M})$. Prove the following.

(i) $|\nu(E)| \leq |\nu|(E)$ for all $E \in \mathcal{M}$.

(ii) $L^1(\nu) = L^1(|\nu|)$, and if $f \in L^1(\nu)$, then $\left|\int f\,d\nu\right| \leq \int |f|\,d|\nu|$.

39. Let ν be a complex measure on $(X, \mathcal{M})$. If $A \in \mathcal{M}$, let

$$\mu_1(A) = \sup\Big\{\sum_1^n \big|\nu(E_j)\big| \;\Big|\; E_1, \ldots, E_n \text{ disjoint, } A = \cup_1^n E_j,\ n \in \mathbf{N}\Big\}$$

$$\mu_2(A) = \sup\Big\{\Big|\int_A f\,d\nu\Big| \;\Big|\; |f| \leq 1\Big\}$$

Prove that $\mu_1 = \mu_2 = |\nu|$. (Hints: to show that $\mu_2 \leq \mu_1$, approximate f by simple functions. To show that $\mu_2 = |\nu|$, let $f = \overline{d\nu/d|\nu|}$.)

40. (i) Let $B(r, x)$ denote the open ball in $\mathbf{R}^n$ with center x and radius r. Prove that the quantity $m\big(B(r,x)\big)$ is a continuous function of (r, x) in $(0, \infty) \times \mathbf{R}^n$.

(ii) Prove that for $f \in L^1_{loc}(\mathbf{R}^n)$, the quantity

$$(*) \qquad \int_{B(r,x)} f\,dm$$

depends continuously on (r, x) in $(0, \infty) \times \mathbf{R}^n$. (Hint: Use the dominated convergence theorem.)