

Solve three problems from among these and past unsolved problems.

31. Let ν be a signed measure on (X, \mathcal{M}) . Prove the following statements. (Use the definition of signed measure, but not the Hahn or Jordan decomposition theorems.)

- (i) If $\{E_j\}$ is an increasing sequence in \mathcal{M} , then $\nu(\cup_j E_j) = \lim_j \nu(E_j)$.
- (ii) If $\{E_j\}$ is a decreasing sequence in \mathcal{M} , and $|\nu(E_1)| < \infty$, then $\nu(\cap_j E_j) = \lim_j \nu(E_j)$.

32. Let ν be a signed measure on (X, \mathcal{M}) . Prove the following statements.

- (i) $L^1(\nu) = L^1(|\nu|)$
- (ii) If $f \in L^1(\nu)$, then $|\int f d\nu| \leq \int |f| d|\nu|$.
- (iii) If $E \in \mathcal{M}$, then $|\nu|(E) = \sup\{|\int_E f d\nu| : |f| \leq 1\}$.

33. Prove the following statements.

- (i) If ν is a signed measure, and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$. (For signed measures θ and ϕ we write $\theta \leq \phi$ to mean that $\theta(E) \leq \phi(E)$ for all measurable sets E .)
- (ii) Let ν_1, ν_2 be signed measures, and suppose that $\nu_1(X)$ and $\nu_2(X)$ are not opposite infinite extended real numbers. Then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

34. Let (X, \mathcal{M}, μ) be a measure space. A collection of functions $\mathcal{E} \subseteq L^1(\mu)$ is called *uniformly integrable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every measurable set E with $\mu(E) < \delta$ and for every $f \in \mathcal{E}$, we have

$$\left| \int_E f d\mu \right| < \epsilon.$$

Prove the following statements.

- (i) Any finite subset of $L^1(\mu)$ is uniformly integrable. (You may cite the result of # 22.)
- (ii) If $f_n \rightarrow f$ in $L^1(\mu)$ then $\{f_1, f_2, f_3, \dots\}$ is uniformly integrable.

35. Let $0 < p < q < \infty$, and let (X, \mathcal{M}, μ) be a measure space.

- (i) Suppose that $\mu(X) < \infty$. Prove that for any measurable function f , if $|f|^q \in L^1$ then $|f|^p \in L^1$.
- (ii) Suppose that μ is counting measure and X is infinite (and $\mathcal{M} = \mathcal{P}(X)$). Prove that for any measurable function f , if $|f|^p \in L^1$ then $|f|^q \in L^1$.
- (iii) Suppose that (X, \mathcal{M}, μ) is the real numbers with Lebesgue measure. Prove that neither implication in part (i) or (ii) holds.