MAT 570 **HW 6** **Due Thursday, 10/6/05**

Solve three problems from among these and past unsolved problems.

26. Suppose that $\mu(X) < \infty$. For f a measurable function, define

$$\rho(f) = \int \frac{|f|}{1 + |f|}\, d\mu.$$

Prove that $d(f, g) = \rho(f - g)$ defines a metric on the space of measurable complex-valued functions on X (where two functions are identified if they agree almost everywhere). Prove also that $d(f_n, f) \to 0$ if and only if $f_n \to f$ in measure.

27. (Lusin's theorem) Let $f : [a, b] \to \mathbf{C}$ be Lebesgue measurable. For $\epsilon > 0$ there exists $E \subseteq [a, b]$ compact such that $m(E^c) < \epsilon$ and $f\big|_E$ is continuous. (Hint: use Egoroff's theorem, and the fact that the continuous functions are dense in L^1.) (Remark: this result seems more interesting when you consider the case of a function on $[0, 1]$ whose set of discontinuities equals $\mathbf{Q} \cap [0, 1]$.)

28. Suppose $f_n \to f$ in measure and $g_n \to g$ in measure. Prove the following:
(i) $f_n + g_n \to f + g$ in measure
(ii) $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.

29. Suppose that μ is a σ-finite measure, and $f_n \to f$ almost everywhere. Prove that there exist measurable sets E_1, E_2, … such that $\bigcup_1^\infty E_j$ is conull and such that $f_n \to f$ uniformly on each E_j.

30. Let $F : \mathbf{R} \to \mathbf{R}$ be increasing and right continuous, with $F(-\infty) = 0$ and $F(+\infty) = 1$. In the equation below, prove that the two integrals are equal in $\overline{\mathbf{R}}$ (m refers to Lebesgue measure):

$$\int_{[0,\infty)} x\, d\mu_F = \int_{[0,\infty)} (1 - F)\, dm.$$