MAT 570 **HW 5** **Due Thursday, 9/29/05**

Solve three problems from among these and past unsolved problems.

21. Let $(\mathbf{X}, \mathcal{M}, \mu)$ be a measure space, and let L^+ be the set of measurable functions $\mathbf{X} \to [0, \infty]$. Let $f \in L^+$, and let $\lambda(E) = \int_E f \, d\mu$ for $E \in \mathcal{M}$. Prove that λ is a measure on $\mathcal{M}$, and that $\int g \, d\lambda = \int fg \, d\mu$ for any $g \in L^+$. (Hint: first suppose that g is simple.)

22. We retain the notations of problem 21. Let $h \in L^+$ be integrable. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $E \in \mathcal{M}$ with $\mu(E) < \delta$ we have $\int_E h \, d\mu < \epsilon$. (Hint: if not, choose very small sets violating the statement, and consider their limsup. Use problem 21.)

23. Let $f \in L^+$ be integrable. Prove that for every $\epsilon > 0$ there exists a measurable set E such that $\mu(E) < \infty$ and $\int_{E^c} f < \epsilon$.

24. Let $f(x) = \begin{cases} 1/\sqrt{x}, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$

Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers, and set

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove the following:

(i) $g \in L^1(\mathbf{R}, m)$.

(ii) g is discontinuous at every real number and is unbounded on every interval, and remains so after any modification on a null set.

(iii) $g^2 < \infty$ a.e., but g^2 is not integrable on any interval.

25. Derive the following formulas by expanding part of the integrand into an infinite series and justifying the term-by-term integration.

(i) For $a > 1$, $\int_0^\infty x^{a-1}(e^x - 1)^{-1} \, dx = \Gamma(a)\zeta(a)$,
where $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} \, dt$ and $\zeta(a) = \sum_{n=1}^\infty n^{-a}$.

(ii) For $a > 1$, $\int_0^\infty e^{-ax} x^{-1} \sin x \, dx = \arctan(a^{-1})$.