MAT 570 **HW 4** **Due Thursday, 9/22/05**

Solve three problems from among these and past unsolved problems.

16. Let $(X, \mathcal{M})$ be a measurable space, and let $f : X \to \overline{\mathbf{R}}$ be a function. Suppose that $\{f > c\} \in \mathcal{M}$ for each rational number c. Prove that f is measurable.

17. Let $(X, \mathcal{M})$ be a measurable space, and let $f_n : X \to \mathbf{C}$ be measurable functions for $n = 1, 2, \ldots$. Prove that the set where $\lim_n f_n$ exists is measurable.

18. Let $f : \mathbf{R} \to \mathbf{R}$ be a function.

(i) Suppose that f is continuous at each point of the complement of a Lebesgue nullset. Prove that f is Lebesgue measurable.

(ii) Suppose that f is continuous from the right at each point of the complement of a Lebesgue nullset. Prove that f is Lebesgue measurable.

19. Let $\mathcal{A}$ be an algebra of subsets of $\mathbf{X}$, and let $\theta : \mathcal{A} \to \mathbf{C}$ be a function satisfying

- $\theta(\emptyset) = 0$.
- $\theta(A \cup B) = \theta(A) + \theta(B)$, whenever $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$.

(Such a function θ is called a *finitely additive complex measure*.) Define another function $\widetilde{\theta} : \mathcal{A} \to [0, \infty]$ by:

$$\widetilde{\theta}(A) = \sup\Big\{\sum_{i=1}^{k} |\theta(E_i)| \;\Big|\; E_1, \ldots, E_k \in \mathcal{A} \text{ are disjoint, } A = \cup_1^k E_i,\ k \in \mathbf{N}\Big\}.$$

(i) Prove that $|\theta(A)| \le \widetilde{\theta}(A)$ for $E \in \mathcal{A}$.

(ii) Prove that $\widetilde{\theta}$ is finitely additive.

20. This problem is concerned with the existence of *Markov measures* on shift space. Let n be a fixed integer greater than 1. Let $X = \prod_{-\infty}^{\infty}\{1, \ldots, n\}$ with the product σ-algebra, $\mathcal{M}$. (In other words, letting $\pi_\ell : X \to \{1, \ldots, n\}$ be the ℓth coordinate projection map, $\mathcal{M}$ is the σ-algebra generated by sets of the form $\pi_\ell^{-1}(j)$, for $\ell \in \mathbf{Z}$ and $1 \le j \le n$.) Define *cylinders* in X by

$$Z_\ell(j_0, j_1, \ldots, j_k) = \bigcap_{r=0}^{k} \pi_{\ell+r}^{-1}(j_r) = \{x \in X : x_\ell = j_0,\ \ldots,\ x_{\ell+k} = j_k\}.$$

Let $A = (a_{ij})$ be an $n \times n$ matrix with non-negative entries such that each row sums to one: $\sum_j a_{ij} = 1$. Let $p = (p_1, \ldots, p_n)$ be a probability vector that is a left eigenvector of A with eigenvalue 1: $p_i \ge 0$, $\sum_i p_i = 1$, $p_j = \sum_i p_i a_{ij}$. Define μ on cylinders by

$$\mu\big(Z_\ell(j_0, \ldots, j_k)\big) = p_{j_0} a_{j_0 j_1} a_{j_1 j_2} \cdots a_{j_{k-1} j_k}.$$

(i) Prove that the above defines a probability measure (also denoted μ) on $\mathcal{M}$.

(ii) Let $\sigma : X \to X$ be the shift map: $\sigma(x)_j = x_{j+1}$. Prove that $\mu \circ \sigma = \mu$.