MAT 570 **HW 3** **Due Thursday, 9/15/05**

Solve three problems from among these and past unsolved problems.

11. Let $F : \mathbf{R} \to \mathbf{R}$ be increasing and right continuous, and let μ be the corresponding Lebesgue-Stieltjes measure on the σ-algebra $\mathcal{M}$ of splitting sets for μ^*. Prove that if $E \in \mathcal{M}$ with $\mu(E) < \infty$ and if $\epsilon > 0$ then there exists a set A equal to a finite union of open intervals such that $\mu(E \Delta A) < \epsilon$. (Hint: use the regularity theorem.)

12. Let m denote Lebesgue measure. Let A be a Lebesgue measurable set.
(i) Suppose that E is the nonmeasurable set constructed by means of the axiom of choice (during lecture in the first week). Prove that if $A \subseteq E$ then $m(A) = 0$.
(ii) Prove that if $m(A) > 0$ then A contains a Lebesgue-nonmeasurable set. (Hint: first reduce to the case where $A \subseteq [0,1)$.)

13. Let A be a Lebesgue measurable set with $m(A) > 0$. Prove that for any $c \in (0,1)$ there exists an open interval I such that $m(A \cap I) > c \cdot m(I)$. (Hint: assume the contrary, and use the definition of Lebesgue outer measure.)

14. If $E \subseteq \mathbf{R}$ let $E - E = \{x - y \mid x \in E, y \in E\}$.
(i) Let $E \in \mathcal{L}$ with $E \subseteq [0,1]$ and $m(E) > 3/4$. Prove that $(-1/2, 1/2) \subseteq E - E$.
(ii) Let $A \in \mathcal{L}$ with $m(A) > 0$. Prove that there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq A - A$. (Hint: use problem 13; then translate and scale until you can use part (i).)

15. Let μ be a measure on $\mathcal{B}_{\mathbf{R}}$. Define the *support* of μ by

$$\operatorname{supp}(\mu) = \mathbf{R} \setminus \Big(\bigcup \big\{ E \mid E \subseteq \mathbf{R},\ E \text{ is open},\ \mu(E) = 0 \big\} \Big).$$

(i) Let $F : \mathbf{R} \to \mathbf{R}$ be increasing and right continuous. Prove that

$$x \in \operatorname{supp}(\mu_F) \iff \forall \delta > 0,\ \exists\, a, b \in (x - \delta, x + \delta) \text{ with } F(a) \neq F(b).$$

(ii) Let $B \subseteq \mathbf{R}$ be a countable set. Prove that there is an increasing right continuous function $F : \mathbf{R} \to \mathbf{R}$ such that $B \subseteq \operatorname{supp}(\mu_F)$ and $\mu_F(\mathbf{R} \setminus B) = 0$. (Hint: if $B = \{c_1, c_2, c_3, \ldots\}$, let $P(x) = \big\{ j \mid c_j \leq x \big\}$, and let $F(x) = \sum \big\{ 2^{-j} \mid j \in P(x) \big\}$.)
(iii) Does there exist a σ-finite measure on $\mathcal{B}_{\mathbf{R}}$ which is not equal to μ_F for any increasing right continuous function $F : \mathbf{R} \to \mathbf{R}$?