MAT 570 **HW 2** **Due Thursday, 9/8/05**

Solve three problems from among these and past unsolved problems.

6. (i) Prove that a finitely additive measure that is continuous from below is countably additive.

(ii) Prove that a finitely additive measure that is finite and continuous from above is countable additive.

7. Let $f : X \to Y$ be a function. Prove the following:

(i) If $\mathcal{N}$ is a σ-algebra on Y then $f^{-1}(\mathcal{N})$ is σ-algebra on X.

(ii) If $\mathcal{M}$ is a σ-algebra on X then $\{A \subseteq Y : f^{-1}(A) \in \mathcal{M}\}$ is a σ-algebra on Y.

(iii) If $\mathcal{E}$ is any non-empty collection of subsets of Y then $f^{-1}\big(\mathcal{M}(\mathcal{E})\big) = \mathcal{M}\big(f^{-1}(\mathcal{E})\big)$.

8. Let $\mathcal{A}$ be an algebra of subsets of X, let $\mathcal{A}_\sigma$ be the collection of countable unions of sets in $\mathcal{A}$, and let $\mathcal{A}_{\sigma\delta}$ be the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ be a premeasure on $\mathcal{A}$ and let μ^* be the outer measure on X induced from μ. Prove the following.

(i) For any $E \subseteq X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$ and $\mu^*(A) \le \mu^*(E) + \epsilon$.

(ii) If $\mu^*(E) < \infty$ then E is a splitting set if and only if there exists $B \subseteq \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$.

(iii) If μ is σ-finite, then (ii) is true for any set $E \subseteq X$.

9. Let μ be a semifinite measure, and let E be a measurable set with $\mu(E) = \infty$. Prove that for any $C > 0$ there is a measurable set F such that $F \subseteq E$ and $C < \mu(F) < \infty$.

10. Let X be the real numbers with the discrete metric: $X = \mathbf{R}$, and $d(x, y) = 1$ if $x \neq y$, $d(x, x) = 0$. Prove that $\mathcal{B}_X \otimes \mathcal{B}_X \neq \mathcal{B}_{X\times X}$. (Hint: let $\mathcal{E} = \{A \times B : A,\ B \subseteq X\}$, and let $\mathcal{A}$ be the collection of all countable disjoint unions of sets in $\mathcal{E}$. Prove that $\mathcal{A}$ is a σ-algebra.)