MAT 570 **HW 12** **Due Tuesday, 12/6/05**

Solve three problems from among these and past unsolved problems.

55. Let $f \in L^2(\mathbf{R})$. (Recall that $(Mf)(x) = xf(x)$.) Prove that $Mf \in L^2$ if and only if $\widehat{f}$ has an L^2-derivative (in the sense of problems # 52-54), and that in this case, $(\widehat{f})' = (2\pi i Mf)\widehat{\ }$. (Hint: use the fact that the Fourier transform is isometric on L^2.)

56. (Quantitative uncertainty principle) Let $f \in L^2(\mathbf{R})$.

(i) Prove that $\|f\|_2^2 \leq 4\pi\|Mf\|_2\|M\widehat{f}\|_2$. (Hint: you might as well assume that the right-hand side is finite. Write $\|f\|_2^2 = \lim_{a\to\infty}\int_{-a}^{a} 1 \cdot f\overline{f}$, and integrate by parts. You will need problems 55 and 53. Show that $\|f'\|_2 = 2\pi\|M\widehat{f}\|_2$.)

(ii) Let a, $b \in \mathbf{R}$. Prove that

$$\int (x-a)^2\big|f(x)\big|^2\,dx \cdot \int (x-b)^2\big|\widehat{f}(x)\big|^2\,dx \geq \frac{\|f\|_2^4}{16\pi^2}.$$

(Hint: let $g(x) = \exp(-2\pi i bx)f(x+a)$.)

57. (Qualitative uncertainty principle)

(i) Prove that if $f \in L^1(\mathbf{R})$ and $f * f = 0$ a.e., then $f = 0$ a.e.

(ii) Let $f \in L^1$, and assume that both f and $\widehat{f}$ have compact support. Prove that $f = 0$ a.e. (Hint: Choose $N > 0$ such that both f and $\widehat{f}$ are supported in $(-N, N)$. Show that $(C_N \cdot M^k f) * f = 0$ a.e. for every integer $k \geq 0$. Use the Weierstrass approximation theorem to approximate C_{-N} by a polynomial on $[-N, N]$. (We use the notation $C_a(x) = \exp(2\pi i a x)$.)

58. Let f_k be the characteristic function of the interval $(-k, k)$, and let $g_k = f_k * f_1$.

(i) Find explicitly the Fourier transform of f_k.

(ii) Find g_k explicitly, and find $\|g_k\|_\infty$.

(iii) Prove that $\|\check{g}_k\|_1 \to \infty$ as $k \to \infty$. (Hint: Use the estimate $\sin 2\pi y \geq \pi y$ for $0 \leq y \leq \frac{1}{2\pi}$ to get a lower bound for $\int_0^{1/2\pi} |\check{g}_k|$.)

(iv) Prove that the Fourier transform $\mathcal{F} : L^1(\mathbf{R}) \to C_0(\mathbf{R})$ is not bounded below. (A linear map T between normed spaces is called *bounded below* if there is a constant $c > 0$ such that $\|T(\xi)\| \geq c\|\xi\|$ for all ξ.)