Solve three problems from among these and past unsolved problems.

52. (Recall the notation for translation of functions on $\mathbf{R}$: $(T_y f)(x) = f(x-y)$, whenever $f : \mathbf{R} \to \mathbf{C}$ and x, $y \in \mathbf{R}$.) Let $f \in L^p(\mathbf{R})$. If there exists $h \in L^p(\mathbf{R})$ such that

$$\lim_{t \to 0} \left\| \frac{1}{t}\big(T_{-t} f - f\big) - h \right\|_p = 0,$$

then h is called the *strong L^p derivative* of f.

Let p and q be conjugate exponents ($1 \le p < \infty$), let $f \in L^p$ have strong L^p derivative h, and let $g \in L^q$. Prove that $f * g$ is a differentiable function, and that $(f * g)' = h * g$.

53. Let $f \in L^p$ ($1 \le p < \infty$), and suppose that f has strong L^p derivative h. Prove that (possibly after modification on a null set) f is absolutely continuous on every bounded interval, and $f' = h$. (Hint: let $g \in C_c(\mathbf{R})$ with $\int g = 1$. Then $f * g_t \to f$ in L^p, and by the previous problem $(f * g_t)' \to h$ in L^p.)

54. Suppose that $f : \mathbf{R} \to \mathbf{C}$ is absolutely continuous on every bounded interval, and that f and f' are both in L^p ($1 \le p < \infty$). Prove that f' is the strong L^p derivative of f. (Hint: write

$$\frac{f(x+t) - f(x)}{t} - f'(x) = \frac{1}{t} \int \big(f'(x+y) - f'(x)\big) \chi_{[0,t]}(y) \cdot \chi_{[0,t]}(y)\, dy,$$

and use Hölder's inequality as in the proof from class of Young's inequality.)