MAT 570 **HW 10** **Due Tuesday, 11/15/05**

Solve three problems from among these and past unsolved problems.

47. When (with proof) does equality hold in the triangle inequality in $L^p(X, \mathcal{M}, \mu)$ if:
(i) $p = 1$?
(ii) $1 < p < \infty$?
(iii)* (Extra Credit) $p = \infty$?

48. (The Vitali Convergence Theorem) Let $1 \leq p < \infty$ and let (f_n) be a sequence in $L^p(X, \mathcal{M}, \mu)$. Prove that (f_n) is Cauchy in L^p if and only if the following three conditions all hold:
(i) $(f_n)_{n=1}^{\infty}$ is Cauchy in measure.
(ii) $\{|f_n|^p \mid n = 1, 2, \dots\}$ is uniformly integrable (see problem 34).
(iii) For all $\epsilon > 0$ there exists $E \in \mathcal{M}$ with $\mu(E) < \infty$ such that

$$\int_{X \setminus E} |f_n|^p \, d\mu < \epsilon \qquad \text{for all } n.$$

(Hint: for the 'if' direction, let A_{mn} be the subset of E where $|f_m - f_n| \geq \epsilon$, E being the set given by (iii). Estimate the integral of $|f_m - f_n|^p$ separately over A_{mn}, $E \setminus A_{mn}$, and $X \setminus E$.)

49. Let $f \in L^p(X, \mathcal{M}, \mu) \cap L^\infty(X, \mathcal{M}, \mu)$ for some $p < \infty$ (then $f \in L^q$ for any $q > p$). Prove that $\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$.

50. Let f be a measurable complex-valued function on $(X, \mathcal{M}, \mu)$. The *essential range* of f is defined to be the set, R_f, of all z in $\mathbf{C}$ such that for every $\epsilon > 0$,

$$\mu\big(\{|f - z| < \epsilon\}\big) > 0.$$

Prove that
(i) R_f is closed.
(ii) $\|f\|_\infty = \sup\{|z| \mid z \in R_f\}$.

51. Let $g \in L^\infty(X, \mathcal{M}, \mu)$. Define a linear operator $T_g : L^p \to L^p$ by $T_g(f) = gf$ $(1 \leq p \leq \infty)$. Prove that $\|T_g\| \leq \|g\|_\infty$, and that equality holds if μ is semifinite.