MAT 570 **HW 1** **Due Thursday, 9/1/05**

Solve three of the following problems.

1. Let $\mathcal{E}_4 = \{[a,b) \mid a < b\}$ and let $\mathcal{E}_5 = \{(-\infty, a) \mid a \in \mathbf{R}\}$. Prove that $\mathcal{M}(\mathcal{E}_4) = \mathcal{M}(\mathcal{E}_5) = \mathcal{B}_{\mathbf{R}}$.

2. Let $\mathcal{M}$ be an infinite σ-algebra.
(a) Prove that $\mathcal{M}$ contains an infinite sequence of disjoint nonempty sets.
(b) Prove that $\mathcal{M}$ is uncountable.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{E_j\}_1^\infty \subseteq \mathcal{M}$. (Recall that $\liminf E_j = \cup_n \cap_{j \geq n} E_j$ and that $\limsup E_j = \cap_n \cup_{j \geq n} E_j$.)
(a) Prove that $\mu(\liminf E_j) \leq \liminf \mu(E_j)$.
(b) Suppose that $\mu(\cup E_j) < \infty$. Prove that $\mu(\limsup E_j) \geq \limsup \mu(E_j)$.

In the next two problems, indicate explicitly whenever you use Zorn's lemma or one of its equivalent formulations.

4. Define an $\mathbf{N}$-*filter* to be a collection $\mathcal{F}$ of nonempty subsets of $\mathbf{N}$ that is closed under the formation of intersections and supersets; i.e. such that (1) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$, and (2) if $A \in \mathcal{F}$, $B \subseteq \mathbf{N}$, and $A \subseteq B$, then $B \in \mathcal{F}$. The set of all $\mathbf{N}$-filters is partially ordered by inclusion. Prove the following:
(i) Every $\mathbf{N}$-filter is contained in a maximal $\mathbf{N}$-filter.
(ii) If $\mathcal{F}$ is a maximal $\mathbf{N}$-filter, and $E \subseteq \mathbf{N}$, then $E \in \mathcal{F}$ or $E^c \in \mathcal{F}$.
(iii) If $\mathcal{F}$ is a maximal $\mathbf{N}$-filter then $\mathcal{F}$ and $\mathcal{P}(\mathbf{N})$ have the same cardinality.

5. Let $\mathcal{C}$ denote the collection of all cofinite subsets of $\mathbf{N}$. Prove the following:
(i) $\mathcal{C}$ is an $\mathbf{N}$-filter. If $\mathcal{F}$ is an $\mathbf{N}$-filter containing $\mathcal{C}$, then every element of $\mathcal{F}$ is an infinite set.
(ii) There is a family $\{S_a : a \in \mathbf{R}\}$ of infinite subsets of $\mathbf{N}$ such that $S_a \cap S_b$ is finite whenever $a \neq b$. (Choose a bijection between $\mathbf{N}$ and $\mathbf{Q}$. For each $a \in \mathbf{R}$ choose a sequence in $\mathbf{Q}$ converging to a.)
(ii) There are uncountably many distinct maximal $\mathbf{N}$-filters containing $\mathcal{C}$. (If $S \subseteq \mathbf{N}$ is an infinite set, let $\mathcal{C}(S)$ be the collection of all supersets of $\{E \cap S : E \in \mathcal{C}\}$.)