Consider the sequence 1, 2, 4, 8, 16, ... If we let a_n denote the nth term of this sequence, we have $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 8$, and so on. With a bit of guesswork we realize that a_n can be given by a formula: $a_n = 2^{n-1}$. Sometimes it isn't so easy to guess a formula for a sequence. What makes it feasible to guess in this instance is the way each term of the sequence is built from the preceding term: $a_2 = 2a_1$, $a_3 = 2a_2$, $a_4 = 2a_3$, etc. In fact, the ellipsis "..." is just an abbreviation for "continue in the same way" — it isn't really useful unless we are able to guess the pattern. A better definition for this sequence would be: let $a_1 = 1$, and for each $n \in \mathbf{N}$ let $a_{n+1} = 2a_n$. A moment's thought will convince you that this does define the sequence, and it doesn't leave us wondering if we are seeing the correct pattern. A definition like this, where the initial term, or terms, of a sequence are specified, and each later term is defined as a function of the preceding terms, is called a *recursive* definition of the sequence. It is precisely the kind of definition that allows us to prove facts about the sequence using induction. For example, if the above sequence is defined recursively by $a_1 = 1$, and $a_{n+1} = 2a_n$ for $n \in \mathbf{N}$, we can use induction to prove that for all $n \in \mathbf{N}$, $a_n = 2^{n-1}$. Here is the proof. When $n = 1$ we have $a_1 = 1 = 2^0 = 2^{1-1}$, so the formula is valid when $n = 1$. Let $n \in \mathbf{N}$, and suppose that the formula is valid for n: $a_n = 2^{n-1}$. Then

$$\begin{aligned} a_{n+1} &= 2a_n, \text{ by the recursive definition,} \\ &= 2(2^{n-1}), \text{ by the inductive hypothesis,} \\ &= 2^n \\ &= 2^{(n+1)-1}. \end{aligned}$$

By induction, the formula is true for all n.

Another very important example of a sequence defined recursively is the factorial. The usual definition of *n-factorial* is: $n! = 1 \cdot 2 \cdot 3 \cdots n$. As we saw above, the $\cdots$ is hiding a recursive definition. Here it is:

$$1! = 1, \text{ and for } n \in \mathbf{N},\ (n+1)! = (n+1) \cdot n!.$$

Here is an example using factorial: prove that for $n \geq 4$, $n! > 2^n$. For the proof, first consider the case $n = 4$:

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24 > 16 = 2^4.$$

Now let $n \geq 4$ and suppose that $n! > 2^n$. We have

$$\begin{aligned} (n+1)! &= (n+1)n! \\ &> (n+1)2^n \\ &> 2 \cdot 2^n, \text{ since } n+1 \geq 4+1 = 5 > 2, \\ &= 2^{n+1}. \end{aligned}$$

Many innocent-seeming induction problems have a recursively defined sequence hidden inside. For example, consider the formula

$$1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n+1).$$

The $\cdots$ on the left side is hiding a recursion. Let's make it explicit. Define a sequence a_n by $a_1 = 1$, and for $n \in \mathbf{N}$, $a_{n+1} = a_n + n + 1$. This captures precisely what the left-hand side is merely suggesting. This problem is really claiming that if a_n is defined recursively as above, then $a_n = \frac{1}{2}n(n+1)$. If you look back at the proof of this formula (the first formula that everyone proves with induction), you will see that the recursive definition of the left-hand side is what is actually used.

As a final example of a recursively defined sequence, we present the Fibonacci numbers. They are defined by: $F_1 = 1$, $F_2 = 1$, and for $n \geq 2$, $F_{n+1} = F_n + F_{n-1}$. Notice that the recursion involves *two* preceding terms rather than just one — because of this we have to give the first two terms explicitly. The first few Fibonacci numbers are: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... The Fibonacci numbers have many beautiful properties that can be proved using induction. Often one has to use the later variants of induction (e.g. Theorems 6.3.8 and 6.3.9 in the text), but not always. Here is an example.

Theorem. For $n \geq 2$, $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$.

To prove it, first consider the case $n = 2$:

$$F_2^2 - F_3F_1 = (1)^2 - (2)(1) = 1 - 2 = -1 = (-1)^{2-1}.$$

Now let $n \geq 2$ and suppose that $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$. We have

$$\begin{aligned}
F_{n+1}^2 - F_{n+2}F_n &= F_{n+1}^2 - (F_{n+1} + F_n)F_n, \text{ by the recursive definition of } F_{n+2}, \\
&= F_{n+1}^2 - F_{n+1}F_n - F_n^2 \\
&= F_{n+1}(F_{n+1} - F_n) - F_n^2 \\
&= F_{n+1}F_{n-1} - F_n^2, \text{ by the recursive definition of } F_{n+1}, \\
&= -(F_n^2 - F_{n+1}F_{n-1}) \\
&= -(-1)^{n-1}, \text{ by the inductive hypothesis,} \\
&= (-1)^n \\
&= (-1)^{(n+1)-1}.
\end{aligned}$$

F. Problems involving factorials

1. Prove that for $n \geq 9$, $n! \geq (2^n)^2$.

2. Prove that for $n \in \mathbf{N}$, $n! \leq 2^{(n^2)}$.

3. Let $a_1 = 0$, and for $n \in \mathbf{N}$ let $a_{n+1} = a_n + n \cdot n!$. Find a formula for a_n, and prove that it is correct.

4. For $n \in \mathbf{N}$, $\dfrac{(2n)!}{(n!)^2} \leq 4^n$.

5. For $n \in \mathbf{N}$, $\dfrac{(2n)!}{(n!)^2} \geq 3^n$.