

1. Which of the following statements are equivalent to each other?

$$\begin{array}{lll} P \rightarrow Q & \neg P \rightarrow \neg Q & \neg(P \rightarrow Q) \\ Q \rightarrow P & P \rightarrow \neg Q & \neg(Q \rightarrow P) \\ \neg P \wedge Q & Q \rightarrow \neg P & P \wedge Q \\ \neg Q \vee P & \neg(P \wedge \neg Q) & P \vee Q \end{array}$$

2. Let  $n$  be an integer. Prove that if  $n^2$  is divided by 4 then the remainder is either 0 or 1.

3. Prove that for every positive real number  $x$ , there exists a real number  $y$  such that  $y(y+1) = x$ .

4. Prove that for every real number  $x$ ,  $|2x-3| > x$  if and only if  $|x-2| > 1$ .

5. Prove that for every real number  $x$ ,  $|x-3| < x$  if and only if  $x > 3/2$ .

6. Prove that if  $x$  and  $y$  are rational numbers, then  $x-y$  is a rational number.

7. Prove that if  $x$  is a nonzero rational number, and if  $y$  is an irrational number, then  $xy$  is an irrational number.

8. Decide if the statement is true or false, and prove your answer:

(i) For every real number  $x$ , there exists a real number  $y$ , such that  $x+y^2 > 1$ .

(ii) There exists a real number  $y$  such that for every real number  $x$ ,  $x+y^2 > 1$ .

(iii) For every real number  $y$ , there exists a real number  $x$ , such that  $x+y^2 > 1$ .

(iv) There exists a real number  $x$  such that for every real number  $y$ ,  $x+y^2 > 1$ .

9. Let  $x, y \in \mathbf{R}$ . Suppose that  $x-y \notin \mathbf{Q}$ . Prove that  $\{x+z \mid z \in \mathbf{Q}\}$  and  $\{y+z \mid z \in \mathbf{Q}\}$  are disjoint.

10. Prove that  $A - (A \cap B) = A - B$ .

11. Prove that  $(A \cup B) - B \subseteq A$ .

12. Suppose that  $A \cap C \subseteq B$ . Prove that  $C$  and  $A - B$  are disjoint.

13. Suppose that  $A \subseteq B - C$ , and that  $A$  is non-empty. Prove that  $B \not\subseteq C$ .

14. Prove that  $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$ .

15. Prove that  $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ .

16. Let  $\{A_i \mid i \in I\}$  be a family of sets, and let  $B$  be a set. Prove that  $\bigcup_{i \in I} A_i \subseteq B$  if and only if  $\{A_i \mid i \in I\} \subseteq \mathcal{P}(B)$ .

17. Prove that  $\{n^2 - n \mid n \in \mathbf{Z}\} = \{n^2 + n \mid n \in \mathbf{Z}\}$ .

18. Find all instances of  $A \subseteq B$ , where  $A$  and  $B$  come from the following list of sets:

$$\begin{array}{ll} \{\emptyset\} & \emptyset \\ 2 & \mathbf{N} \\ \{1, 2, 3\} & \mathcal{P}(\emptyset) \\ \{\{1, 2\}\} & \mathcal{P}(\mathbf{N}) \end{array}$$

19. Is  $\{(x^2, x^3) \mid x \in \mathbf{R}\}$  a function from  $[0, \infty)$  to  $\mathbf{R}$ ? Prove your answer

20. Is  $\{(x^2, x^4) \mid x \in \mathbf{R}\}$  a function from  $[0, \infty)$  to  $\mathbf{R}$ ? Prove your answer.

21. Is the following a function from  $\mathbf{R}$  to  $\mathbf{R}$ ? Prove your answer.

$$\{(x, 0) \mid |x| \geq 1\} \cup \{(x, 1+x) \mid -1 \leq x \leq 0\} \cup \{(x, 1-x) \mid 0 \leq x \leq 1\}.$$

22. Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be given by  $f(x) = x^2$ . Find the following sets:

(i)  $f_*([-3, 2])$

(ii)  $f^*([-3, 2])$

(iii)  $f_*([0, \infty))$

(iv)  $f^*([0, \infty))$

23. Let  $g : \mathbf{R}^2 \rightarrow \mathbf{R}$  be given by  $g(x, y) = x^2 + y^2$ . Find the following sets:

(i)  $g_*([0, 1] \times [0, 1])$

(ii)  $g^*([0, 1])$

(iii)  $g_*([1, 2] \times [1, 2])$

(iv)  $g_*((2, 3))$

- 24.** Let  $f : A \rightarrow B$ , and let  $S \subseteq A$  and  $T \subseteq B$ .  
 (i) Suppose that  $f$  is injective. Prove that if  $T \subseteq f_*(S)$  then  $f^*(T) \subseteq S$ .  
 (ii) Suppose that  $f$  is surjective. Prove that if  $f^*(T) \subseteq S$  then  $T \subseteq f_*(S)$ .
- 25.** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ .  
 (i) Prove that if  $g \circ f$  is injective then  $f$  is injective.  
 (ii) Prove that if  $g \circ f$  is surjective then  $g$  is surjective.
- 26.** Prove that  $\frac{x}{1+x}$  defines a bijective function from  $(-1, \infty)$  to  $(1, \infty)$ .
- 27.** Prove that there is a bijective function from  $\mathbf{N}$  to  $\mathbf{Z}$ .
- 28.** Prove that there is a bijective function from  $[-2, 3]$  to  $[2, 9]$ .
- 29.** Prove that there is a bijective function from  $[0, 2)$  to  $(0, 1]$ .
- 30.** Prove that there is a bijective function from  $(0, 1]$  to  $[1, \infty)$ .
- 31.** Prove that  $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$  for all  $n \in \mathbf{N}$ .
- 32.** Prove that  $2^n > n^3$  for  $n \geq 10$ .
- 33.** Let  $a_0 = 0$ , and define  $a_{n+1} = 2a_n + n$  for  $n \in \mathbf{Z}$  with  $n \geq 0$ . Prove that  $a_n = 2^n - n - 1$  for  $n \geq 0$ .
- 34.** Let  $a_0 = 0$ ,  $a_1 = 1$ , and  $a_n = 5a_{n-1} - 6a_{n-2}$  for  $n \geq 2$ . Prove that  $a_n = 3^n - 2^n$  for  $n \geq 0$ .
- 35.** Let  $a \in \mathbf{R}$  with  $a > -1$ . Prove that for all  $n \in \mathbf{N}$ ,  $(1 + a)^n \geq 1 + na$ .
- 36.** Prove that for  $n \in \mathbf{N}$ ,  $n! \leq 2^{(n^2)}$ .
- 37.** Let  $a_1 = 0$ , and for  $n \in \mathbf{N}$  let  $a_{n+1} = a_n + n \cdot n!$ . Find a formula for  $a_n$ , and prove that it is correct.
- 38.** Let  $A = \{1, 2, 3\}$ .  
 (i) Give an example of a relation on  $A$  that is reflexive and transitive, but not symmetric.  
 (ii) Give an example of a relation on  $A$  that is reflexive and symmetric, but not transitive.  
 (iii) Give an example of a relation on  $A$  that is symmetric and transitive, but not reflexive.
- 39.** In each part decide if the given relation is reflexive, symmetric, and/or transitive. Describe the indicated relation classes.  
 (i)  $R$  is the relation on  $\mathbf{R}^2$  defined by:  $(x, y)R(z, w)$  iff  $x + y = z + w$ . Find  $[(0, 0)]$  and  $[(1, 1)]$ .  
 (ii)  $S$  is the relation on  $\mathbf{N}^2$  defined by:  $(a, b)S(c, d)$  iff  $ad = bc$ . Find  $[(1, 2)]$  and  $[(3, 3)]$ .  
 (iii)  $T$  is the relation on  $\mathcal{P}(\mathbf{N})$  defined by:  $ETF$  iff  $E \supseteq F$ . Find  $[\{1, 3\}]$  and  $[\emptyset]$ .  
 (iv)  $U$  is the relation on  $(0, \infty)$  defined by:  $xUy$  iff  $\frac{x}{y} \in \mathbf{Z}$ . Find  $[1]$ .
- 40.** Solve the equation  $[4] \cdot x = [2]$  in  $\mathbf{Z}_6$ .
- 41.** Solve the equation  $[4] \cdot x = [2]$  in  $\mathbf{Z}_7$ .
- 42.** Solve the equation  $x^2 = [-1]$  in  $\mathbf{Z}_7$ .
- 43.** Solve the equation  $x^2 = [-1]$  in  $\mathbf{Z}_{13}$ .
- 44.** Let  $R$  be a relation on the set  $A$ , and suppose that for all  $x, y \in A$ , if  $xRy$  then  $[y] \subseteq [x]$ . Prove that  $R$  is transitive.

### Definitions for Review

**Chapter 1.** and, or, negation, conditional, biconditional, universal and existential quantifiers

**Chapter 2.** even integer, odd integer, divides, prime number, rational number, irrational number, absolute value

**Chapter 3.**  $A \subseteq B$ ,  $A \cup B$ ,  $A \cap B$ ,  $A - B$ , disjoint sets,  $A \times B$ ,  $\mathcal{P}(A)$ ,  $\bigcup_{i \in I} A_i$ ,  $\bigcap_{i \in I} A_i$

**Chapter 4.**  $f : A \rightarrow B$ ,  $f(x)$ ,  $1_A$ ,  $f_*(S)$ ,  $f^*(T)$ ,  $f \circ g$ , injective function, surjective function, bijective function, left inverse, right inverse, inverse

**Chapter 5.** relation, relation class, reflexive relation, symmetric relation, transitive relation, congruence modulo  $n$ , equivalence relation, partition

**Chapter 6.** finite set, infinite set, countably infinite set, countable set, uncountable set, Fibonacci numbers, factorial