

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions. **Reread** your proofs **before** copying them out to turn in; I really do mean that you should write (at least) one draft of each solution. If you submit some nonsense in the course of a proof, the whole thing might strike me as not worth reading.

49. In all parts of this problem, prove that your answer is correct.

- (i) Find a sequence of continuous functions $f_n : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow 0} f_n(x) \neq \lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} f_n(x),$$

and such that both iterated limits exist. Show that this is impossible if (f_n) converges pointwise to a function that is continuous at 0.

- (ii) Find a sequence of continuous functions $f_n : [0, 1] \rightarrow \mathbf{R}$ such that f_n converges to 0 pointwise, the sequence of integrals $\int_0^1 f_n$ is bounded, and the sequence of integrals $\int_0^1 f_n$ does not converge.

50. In all parts of this problem, prove that your answer is correct.

- (i) Find a sequence of Riemann integrable functions on $[0, 1]$ that converge to a bounded function that is not Riemann integrable.
- (ii) Find a uniformly convergent sequence of differentiable functions $f_n : \mathbf{R} \rightarrow \mathbf{R}$ such that the sequence (f'_n) diverges on a dense subset of \mathbf{R} . Can you arrange that (f'_n) diverges at every point of \mathbf{R} ?

51. Let (a_n) be a sequence of non-zero real numbers. Prove that

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} \leq \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

(Remark: In particular, this shows that the root test implies convergence whenever the ratio test does.)

52. For a real number x let (x) denote the *fractional part* of x : $(x) = x - [x]$, where $[x]$ is the greatest integer function. Define $f : \mathbf{R} \rightarrow \mathbf{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}.$$

- (i) Prove that f is continuous on $\mathbf{R} \setminus \mathbf{Q}$.
- (ii) Prove that f is discontinuous on \mathbf{Q} .
- (iii) Prove that f is Riemann integrable on $[a, b]$, for any compact interval $[a, b]$.