MAT 578 **HW 2** **Due Wednesday, 9/10/03**

Choose three of the following problems.

5. Let X and Z be inner product spaces, and let $T : X \to Z$ be a function.

(i) Suppose that T preserves inner products. Prove that T is linear and preserves norms.

(ii) Supposet that T is linear and preserves norms. Prove that T preserves inner products.

6. Let $\big(X, \langle\cdot,\cdot\rangle\big)$ be an inner product space. For elements $x_1, \ldots, x_n \in X$ let $M(x_1, \ldots, x_n)$ denote the $n \times n$ matrix whose (i, j)-entry is $\langle x_i, x_j\rangle$. Let $G(x_1, \ldots, x_n)$ denote the determinant of $M(x_1, \ldots, x_n)$.

(i) Prove that $G(x_1, \ldots, x_n) \geq 0$. (Hint: use an orthonormal basis for the subspace spanned by $x_1, \ldots, x_n$ to identify $M(x_1, \ldots, x_n)$ as a self-adjoint matrix with non-negative eigenvalues.)

(ii) Prove that $G(x_1, \ldots, x_n) = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent.

(iii) Prove that $G(y, x_1, \ldots, x_n)/G(x_1, \ldots, x_n)$ equals the square of the distance from y to $\operatorname{span}\{x_1, \ldots, x_n\}$.

7. Let $H_1, H_2, \ldots$ be a sequence of Hilbert spaces. Let $H = \big\{x = (x_1, x_2, \ldots) : x_i \in H_i \text{ and } \sum_i \|x_i\|^2 < \infty\big\}$.

(i) Let $x, y \in H$. Prove that $\sum_i \langle x_i, y_i\rangle$ is absolutely convergent.

Let $\langle x, y\rangle$ denote the sum in part (i). Note that this is an inner product on H.

(ii) Prove that $\big(H, \langle\cdot,\cdot\rangle\big)$ is a Hilbert space.

(iii) For each i let $\mathcal{B}_i$ be a basis for H_i. Prove that $\mathcal{B} = \cup_i \mathcal{B}_i$ is a basis for H.

8. Let H be an infinite dimensional Hilbert space. Prove that there is a linear functional on H which is not continuous. (Hint: use a Hamel basis for H.) Conclude that H contains a dense linear manifold of codimension one. (If W is a vector subspace of V then the *codimension* of W in V is defined to be the dimension of the quotient vector space V/W.)