MAT 578 **HW 14** **Due Monday, 12/8/03**

Choose three from among the following problems and any as-yet unworked problems in previous assignments.

50. Let H be a Hilbert space. We will let $\mathcal{U}(H)$ denote the group of unitary operators on H. Let S be the unilateral shift on ℓ^2.

(i) Prove that $S^n \to 0$ in the weak operator topology, but not in the strong operator topology.

(ii) Prove that there exist $U_n \in \mathcal{U}(\ell^2)$ such that $U_n \to S$ in the strong operator topology.

(iii) Suppose V, $V_n \in \mathcal{U}(H)$ and $V_n \to V$ in the weak operator topology. Prove that $V_n \to V$ in the strong operator topology.

51. Let $\mathcal{P}(H)$ denote the set of projections in $B(H)$.

(i) Prove that $\mathcal{P}$ is closed in the strong operator topology (and hence also in the norm topology).

(ii) Suppose that H is infinite dimensional. Prove that $\mathcal{P}(H)$ is not closed in the weak operator topology.

52. (i) Let $(A_n)_{n=1}^{\infty}$ be a monotone, bounded sequence of self adjoint operators in $B(H)$. (I.e., either $A_1 \leq A_2 \leq \cdots$ or $A_1 \geq A_2 \geq \cdots$, and for some constant $C > 0$, $\|A_n\| \leq C$ for all n.) Prove that there exists a self adjoint operator $A \in B(H)$ such that $A_n \to A$ in the strong operator topology. (Hint: define A by $\langle Ax, x\rangle$ and polarization.)

(ii) Let P and Q be projections in $B(H)$. We let $P \wedge Q$ denote the projection onto $PH \cap QH$. Prove that $(PQ)^n \to P \wedge Q$ in the strong operator topology. (Hint: show that $(PQP)^n$ is a decreasing sequence of positive operators.)

53. Let $T \in B(H)$. Prove that T is compact if and only if TH does not contain a closed infinite dimensional subspace of H.