

Choose three from among the following problems and any as-yet unworked problems in previous assignments.

34. Let X be an infinite dimensional Banach space.

- (i) Prove that a Hamel basis for X must be uncountable.
- (ii) Prove that the weak topology of X is not metrizable. (Hint: show that the weak topology cannot have a countable neighborhood base at the origin.)

35. Let X be an infinite dimensional Banach space. Let $B = \{x \in X : \|x\| \leq 1\}$ be the closed unit ball, and let $S = \{x \in X : \|x\| = 1\}$ be the unit sphere.

- (i) Prove that B is weakly closed.
- (ii) Prove that S is not weakly closed.
- (iii) Find the weak closure of S .

36. For $1 \leq m < n$ let x_{mn} be the sequence

$$x_{mn}(j) = \begin{cases} 1, & \text{if } j = m \\ m, & \text{if } j = n \\ 0, & \text{otherwise.} \end{cases}$$

Let $E = \{x_{mn} \mid 1 \leq m < n\} \subseteq \ell^2$. Prove that 0 is in the weak closure of E , but that no sequence in E converges weakly to 0.

37. Let ℓ^∞ be the Banach space of all bounded complex sequences, with the supremum norm. For each $n \geq 1$ define $\phi_n \in (\ell^\infty)^*$ by

$$\phi_n(x) = \frac{1}{n} \sum_{j=1}^n x_j.$$

Let $E = \{\phi_1, \phi_2, \dots\} \subseteq (\ell^\infty)^*$. Prove that

- (i) E has a weak* cluster point in $(\ell^\infty)^*$.
- (ii) No weak* cluster point of E lies in $\kappa(\ell^1)$.
- (iii) Let $T : \ell^\infty \rightarrow \ell^\infty$ be the shift map: $(Tx)_j = x_{j+1}$. If ϕ is a weak* cluster point of E , then $\phi \circ T = \phi$.