MAT 472 **HOMEWORK 2** **Due: 9/10/02**

Homework is due in my mailbox by 3:00. Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions. **Reread** your proofs **before** copying them out to turn in; I really do mean that you should write (at least) one draft of each solution. If you submit some nonsense in the course of a proof, the whole thing might strike me as not worth reading.

6. Recall the function d on the space $X = \prod_{n=1}^{\infty}\{0,\, 1\}$ defined in class: for $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ distinct elements of X, we put $k(x,y) = \min\{i \in \mathbf{N} \mid x_i \neq y_i\}$, and let

$$d(x,y) = \begin{cases} 1/k(x,y), & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

Let x, $y \in X$ and $r > 0$.

(i) Prove that if $d(x,y) \geq r$, then $B_r(x) \cap B_r(y) = \emptyset$.

(ii) Prove that if $d(x,y) < r$, then $B_r(x) = B_r(y)$.

(Hint: if $0 < r \leq 1$, then there exists $n \in \mathbf{N}$ with $1/(n+1) < r \leq 1/n$.)

7. Define $\|\cdot\|_1$ on $\mathbf{R}^2$ by:

$$\|x\|_1 = |x_1| + |x_2|.$$

(i) Prove that $\|\cdot\|_1$ is a norm on $\mathbf{R}^2$.

(ii) Sketch the open unit ball with center $(0,0)$ for the metric d_1 associated to $\|\cdot\|_1$. (No proof needed.)

(iii) Let d be the usual (Euclidean) metric on $\mathbf{R}^2$. Prove that each open ball for the metric d is an open set for the metric d_1.

8. Prove that the set $\{x \in \mathbf{R}^2 \mid x_1 x_2 = 1 \text{ and } x_1 > 0\}$ is a closed set.

9. Let $\mathcal{B}$ be the collection of open balls in $\mathbf{R}^n$ having rational radius and having a center with all coordinates rational.

(i) Prove that $\mathcal{B}$ is a countable collection.

(ii) Prove that if U is any open ball in $\mathbf{R}^n$, and if x is any point of U, then there is a ball $V \in \mathcal{B}$ such that $x \in V$ and $V \subseteq U$.

10. (i) Prove that $\overline{B_r(a)} = \overline{B}_r(a)$ in $\mathbf{R}^n$.

(ii) Does the same result hold in the space X of problem 6? Prove your answer.