MAT 472 **HOMEWORK 10** **Due: Tuesday, 11/26/02**

Homework is due in my mailbox by 3:00. Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions. **Reread** your proofs **before** copying them out to turn in; I really do mean that you should write (at least) one draft of each solution. If you submit some nonsense in the course of a proof, the whole thing might strike me as not worth reading.

42. Let (a_n) be a sequence of non-zero real numbers. Prove that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \le \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(Remark: In particular, this shows that the root test implies convergence whenever the ratio test does.)

43. (i) Let (a_n) be a sequence of non-zero real numbers. Suppose that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1. \quad \text{Let } c = \limsup_{n\to\infty}\ n \cdot \left(\left|\frac{a_{n+1}}{a_n}\right| - 1\right).$$

Prove that $\sum a_n$ converges absolutely if $c < -1$.

(Hint: choose $c < \alpha < -1$, let $b_n = n^\alpha$, and consider the limit of $n(b_{n+1}/b_n - 1)$.)

(ii) Let p and q be real numbers with $0 < p < q - 1$. Prove the convergence of

$$\sum_{n=1}^{\infty} \frac{p(p+1)(p+2)\cdots(p+n-1)}{q(q+1)(q+2)\cdots(q+n-1)}.$$

44. Let (a_n) be a sequence of real numbers such that $a_n \neq 0$ for at least one integer n. Suppose that the power series $\sum a_n x^n$ has a positive radius of convergence, and let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for x in the interval of convergence. Prove that there exists $\delta > 0$ such that $f(x) \neq 0$ for all x with $0 < |x| < \delta$.

45. Let (a_n) be a sequence of non-negative real numbers, and suppose that $\sum a_n$ diverges. In each part below, do one of the following: prove that the series must converge, prove that the series must diverge, or give examples (with proof) showing that the series may converge and may diverge.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+n a_n}$