MAT 578 **HW 7** **Due Thursday, 10/18/01**

Choose three of the following problems.

1. Let X and Y be Banach spaces.

(i) Suppose that $T \in K(X,Y)$. Prove that if (x_n) is a weakly convergent sequence in X, then (Tx_n) is norm convergent in Y.

(ii) Suppose that X is reflexive. Let $T : X \to Y$ be linear, and suppose that (Tx_n) is norm convergent whenever (x_n) is a weakly convergent sequence in X. Prove that T is compact.

2. Find the essential spectrum of the right shift operator on ℓ^p, $1 \leq p < \infty$.

3. Define $T \in B(X,Y)$ to be *left-Fredholm* if there is $S \in B(Y,X)$ such that $ST - 1_X$ is compact. Define *right-Fredholm* similarly.

(i) Prove that T is left-Fredholm if and only if $\ker(T)$ is finite dimensional and TX is closed and complemented in Y.

(ii) State and prove an analogous characterization of right-Fredholm operators.

(iii) Prove that T is Fredholm if and only if T is both left- and right-Fredholm.

4. Let A be a unital commutative Banach algebra, and let G be the group of invertible elements in A. Prove that every element of G has a logarithm if and only if G is connected. (Hint: let U be the open ball centered at the identity with radius one. Prove that every element of the connected component of the identity in G is a finite product of elements of U. (In other words, the subgroup of G generated by U is the connected component of the identity.))