37. Let ν be a signed measure on $(X, \mathcal{M})$. Prove the following statements. (Use the definition of signed measure, but not the Hahn decomposition theorem.)

(i) If $\{E_j\}$ is an increasing sequence in $\mathcal{M}$, then $\nu(\cup_j E_j) = \lim_j \nu(E_j)$.

(ii) If $\{E_j\}$ is a decreasing sequence in $\mathcal{M}$, and $\left|\nu(E_1)\right| < \infty$, then $\nu(\cap_j E_j) = \lim_j \nu(E_j)$.

38. Let μ and ν be signed measures. Prove that the following statements are equivalent.

(i) $\mu \perp \nu$

(ii) $\mu \perp |\nu|$

(iii) $\mu \perp \nu^+$ and $\mu \perp \nu^-$

39. Let ν be a signed measure on $(X, \mathcal{M})$. Prove the following statments.

(i) $L^1(\nu) = L^1(|\nu|)$

(ii) If $f \in L^1(\nu)$, then $\left|\int f\, d\nu\right| \leq \int |f|\, d|\nu|$.

(iii) If $E \in \mathcal{M}$, then $|\nu|(E) = \sup\left\{\left|\int_E f\, d\nu\right| : |f| \leq 1\right\}$.

40. Prove the following statments.

(i) If ν is a signed measure, and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$. (For signed measures θ and ϕ we write $\theta \leq \phi$ to mean that $\theta(E) \leq \phi(E)$ for all measurable sets E.)

(ii) Let ν_1, ν_2 be signed measures, and suppose that $\nu_1(X)$ and $\nu_2(X)$ are not opposite extended real numbers. Then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

41. Let $(X, \mathcal{M}, \mu)$ be a measure space. A collection of functions $\mathcal{E} \subseteq L^1(\mu)$ is called *uniformly integrable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every measurable set E with $\mu(E) < \delta$ and for every $f \in \mathcal{E}$, we have

$$\left|\int_E f\, d\mu\right| < \epsilon.$$

Prove the following statements.

(i) Any finite subset of $L^1(\mu)$ is uniformly integrable. (You may cite the result of # 21.)

(ii) If $f_n \to f$ in $L^1(\mu)$ then $\{f_1, f_2, f_3, \ldots\}$ is uniformly integrable.