Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes
Volume **18**, 1998

Sextic Number Fields with Discriminant $-^j 2^a 3^b$

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Abstract. Complete lists of number fields, of given degree n and unramified outside a given finite set S of primes, are both of intrinsic interest and useful in some applications. For degrees $n \leq 5$ and $S = \{\infty, 2, 3\}$, the complete lists have appeared previously; there are in total 85 such fields. Here we give the complete list for $n = 6$ and $S = \{\infty, 2, 3\}$, finding in particular exactly 398 such fields. We use a three-pronged approach to obtain this classification: an exhaustive computer search, sextic twinning, and class field theory. Also we completely identify the 2-adic and 3-adic completions of all these degree ≤ 6 fields, this information being one of the focal points of interest and essential in applications.

There is a considerable literature on the classification of number fields by means of their discriminants. At present for n = 3, 4, 5, 6, 7 there are large tables, available at [**B**], giving all number fields of degree n and absolute discriminant less than certain bounds.

It would also be interesting to have complete tables of number fields unramified outside a given finite set S of primes. For a given degree n, these sets of number fields are finite, by a classical theorem of Hermite. For a recent survey of the general subject of number fields with prescribed ramification, we recommend [**H**]; there it is made clear that very little is known about the set of non-solvable number fields unramified outside a given S.

In this paper, we focus on the set $S = \{\infty, 2, 3\}$. We present tables for degrees $\leq$ 6, the following chart giving an overview the situation.

Degree n	S_n	A_n	H_n	Upper Bound for $\lvert D\rvert$	Method
2	7	–	–	$2^3 3 = 24$	Trivial
3	8	1	–	$2^3 3^5 = 1{,}944$	Old tables or CFT
4	22	1	39	$2^{11} 3^5 = 497{,}664$	Old tables or CFT
5	5	0	1	$2^{11} 3^6 = 1{,}492{,}992$	Old tables
6	54	8	336	$2^{14} 3^{11} = 2{,}902{,}376{,}448$	New here
7				$2^{14} 3^{11} = 2{,}902{,}376{,}448$	
8				$2^{31} 3^{12} \approx 1.1 \times 10^{15}$	
9				$2^{31} 3^{26} \approx 5.5 \times 10^{21}$	

1991 *Mathematics Subject Classification.* 11Y40.
This is the final form of the paper.

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Here the S_n and A_n columns give the number of fields with the given Galois group. Similarly, the H_n entry is the number of fields with smaller Galois group.

For $n \leq 4$, the maximal possible absolute discriminant is well within the range of the existing tables; alternatively, one can use class field theory, as all fields here are solvable. The case $n = 5$ is just barely within the range of the existing tables. For $n = 6$, the existing tables for arbitrary signature fields cover absolute discriminants up to 200,000; the maximum possible absolute discriminant required here is well beyond the range of these tables. The case $n = 7$ may also be in the range of our current techniques, while the cases $n \geq 8$ certainly are not.

In general, we would like to advocate the point of view that number fields with discriminants divisible by only a few small primes should be regarded as "jewels". On the one hand, they occur very rarely on complete tables of number fields ordered by absolute discriminant. On the other hand, they are exactly the fields which appear most often in certain other contexts. For example, they appear as fields of definition of three-point covers of the projective line, see e.g. [**Mal**]. Similarly, they arise naturally in the study of motives. Suppose M is a motive with bad reduction only in S and ℓ is a finite prime in S; then the fields which correspond to the ℓ-adic representations associated to M are ramified only in S. We have already investigated some simple cases from quite varied geometric situations with $S = \{\infty, 2, 3\}$. The fields we tabulate here appear repeatedly. Our tables thus play an important role in the analysis of certain motives. This application served as our principal motivation for writing the present paper.

To obtain the complete classification of sextics, we proceed as follows. There are sixteen conjugacy classes of transitive subgroups of S_6. We divide them into three classes:

- Two "new" non-solvable groups: A_6 and S_6
- Two "new" solvable groups: $C_3^2.C_4$ and $C_3^2.D_4$
- Twelve "old" groups.

We have made a large specialized computer search for fields of discriminant $-^j 2^a 3^b$, in particular guaranteed to find all the new non-solvable fields. The search, described in Section 2, found 62 new non-solvable fields, 54 new solvable fields and 282 old fields. In Section 3.2, we use sextic twinning to directly construct all old fields from the list of degree ≤ 5 fields; this proves that there are indeed 282 of them. In Section 3.4, we apply class field theory to the list of C_4 and D_4 quartics, proving directly that there are indeed 54 new solvable fields.

The fields we consider here being "jewels", it is worth studying them in some detail. We tabulate essentially all such fields of degree ≤ 5 and all of the new sextic fields. These tables provide common invariants for each field, and also complete descriptions of their 2-adic and 3-adic completions.

1. Preliminaries

1.1. Conventions. A number field is by definition a finite degree field extension of $\mathbb{Q}$. Similarly, a p-adic field is a finite degree field extension of $\mathbb{Q}_p$. A number algebra is a finite direct product of number fields and a p-adic algebra is a finite direct product of p-adic fields. While the focus is on fields, it is natural, as often in Galois theory, to consider algebras as well. For example, if K is a number field its p-adic completions K_p are typically only p-adic algebras.

Throughout, when the context is clear, we will drop phrases such as "isomorphism classes of" and "up to isomorphism". For example, when we say there are exactly 398 sextic number fields with discriminant $-^j 2^a 3^b$, we of course mean 398 isomorphism classes of such fields.

With respect to Galois theory, we work with $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We fix also algebraic closures $\overline{\mathbb{Q}}_2 \supset \overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_3 \supset \overline{\mathbb{Q}}$. So we have unambiguously defined decomposition groups $D_v := \mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$ for $v = \infty$, 2, and 3.

If K is a number algebra, its root set is by definition $X := \mathrm{Hom}(K, \overline{\mathbb{Q}})$. The construction $K \mapsto X$ gives an antiequivalence between the category of number algebras and the category of finite $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-sets, number fields corresponding to transitive $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-sets. The Galois group of K is by definition the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in the symmetric group S_X. Thus, for example, for a sextic number field K, one often has $\mathrm{Gal}(K) \cong S_6$. Similarly, one has decomposition groups $\mathrm{Gal}(K_v) \subseteq \mathrm{Gal}(K)$.

We do our best to consider the place ∞ of $\mathbb{Q}$ on an equal footing with the remaining places 2, 3, ... Accordingly, for us, the discriminant of a number algebra K is a formal symbol $-^j 2^a 3^b \cdots$. Here j is the number of complex places of K and a, b, ... are as usual. A place is ramified in K iff the corresponding exponent is positive.

The discriminant class d of a number algebra K is its discriminant considered as an element of $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. One can think of the discriminant class of a number field as just its discriminant with all exponents considered in $\mathbb{Z}/2$; equivalently, one can think of a discriminant class as just a square-free integer.

The absolute discriminant of a number algebra K of discriminant D_K is just the positive integer $|D_K|$. We then define the root discriminant to be $\mathrm{rd}(K) := |D_K|^{1/[K:\mathbb{Q}]}$ and the Galois root discriminant of K to be the root discriminant of a splitting field K^{gal}, i.e. $\mathrm{grd}(K) := \mathrm{rd}(K^{\mathrm{gal}})$.

1.2. Guide to the tables. Here we explain in some detail the format of Table 1. The main table of this paper, Table 6, has essentially the same format. The few differences will be explained there.

The possible Galois groups are listed on the second line of the following table.

	$n=3$		$n=4$					$n=5$				
G	S_3	A_3	S_4	A_4	D_4	C_4	V	S_5	A_5	F_5	D_5	C_5
#	8	1	22	1	28	4	7	5	0	1	0	0

Here F_5 denotes the Frobenius group $\mathbb{F}_5.\mathbb{F}_5^\times$ of order 20; the other notations are standard. The third line lists the number of $-^j 2^a 3^b$ fields with the given Galois group and degree. Table 1 omits the twenty-eight D_4 fields and the four C_4 fields; they will be tabulated along with the sextics in Table 6, where they will play a useful role as resolvents. Table 1 also omits the seven V fields; they are trivial from the point of view of classification, being naturally indexed by four-element subgroups of $\{-6, -3, -2, -1, 1, 2, 3, 6\}$.

There is a subtable for each discriminant class $d \in \{-6, -3, -2, -1, 1, 2, 3, 6\}$. After the subtable name, there are two lines of header. If $d \neq 1$, there are two lines corresponding to the quadratic field $D = \mathbb{Q}[x]/(x^2 - d)$. Finally, there are two lines for each of the fields K being tabulated.

Our policy throughout is to print invariants of K so that the various aspects of our situation are each presented as clearly as possible. There are many relationships among the invariants. Often some even completely determine others, as we'll indicate. Thus we have aimed for maximum clarity at the expense of some redundancy.

Galois groups, Artin exponents, and root numbers. Let K be one of our listed fields. The entry G gives the Galois group of K, together with some information about lifting explained below. The entries in the slots ∞, 2, 3 are of the form $(c_v)_{w_v}$. Here c_v is the Artin exponent at v; thus the discriminant of K is $-^{c_\infty} 2^{c_2} 3^{c_3}$. Also, w_v is the root number at v, a fourth root of unity in $\mathbb{C}$. To save space and increase readability we use the notation

$$\begin{array}{cccccc} + & \text{for} & 1 & - & \text{for} & -1 \\ i & \text{for} & i & k & \text{for} & -i. \end{array}$$

The root number $w_v(K)$ differs from the root number $w_v(D)$ of the quadratic discriminant algebra D by a sign. Also, one always has $w_\infty w_2 w_3 = 1$. The root number w_∞ is just k^{c_∞}, so in this spot one sees only the symbols 0_+, 1_k and 2_-; the other root numbers are more subtle to calculate.

Quadratic lifting. For $n \geq 4$, the group A_n has up to isomorphism a unique non-split double cover $\widetilde{A}_n$. The group S_n has two similarly well-defined double covers restricting to $\widetilde{A}_n$. There are several notations for these covers in the literature, some conflicting. We let $\widetilde{S}_n$ be the double cover where 2-cycles in S_n lift to elements of order two; we let $\widehat{S}_n$ be the double cover where 2-cycles in S_n lift to elements of order 4. Given a degree n number algebra K with root set $X = \mathrm{Hom}(K, \overline{\mathbb{Q}})$, the permutation representation $\rho_K \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to S_X$ may or may not lift to $\widetilde{S}_X$; similarly, it may or may not lift to $\widehat{S}_X$.

Lifting criteria in two related languages are given in [**S**] and [**D**]. Very simply,

$$\rho_K \text{ lifts to a } \tilde{\rho}_K \iff \forall v,\ w_v(K) = w_v(D)$$
$$\rho_K \text{ lifts to a } \hat{\rho}_K \iff \forall v,\ w_v(K) = w_v(D)(d,-1)_v$$

Here $(d, -1)_v \in \{\pm 1\}$ is the Hilbert sign. For convenience, on the discriminant class d-table we have put the header v in bold-face iff $(d, -1)_v = -1$. If K has Galois group G, we write G, $\widetilde{G}$, $\widehat{G}$ or $\overline{G}$ to indicate whether one has no lifting, tilde-lifting only, hat-lifting only, or both liftings. Perhaps the most interesting fact about the seven omitted V quartics is the following. Since V is a subgroup of the alternating group A_4, the tilde and hat obstructions coincide for each place v; they all vanish only for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, and so this field only is contained in quaternionic octic fields. In fact, $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is contained in two such $-^j 2^a 3^b$ fields, one totally real and the other totally imaginary.

Slope content. In the slot SC_p, we give the slope content of K_p at p. By this we mean the following. The decomposition group $D_p = \mathrm{Gal}(K_p)$ has a decreasing filtration by ramification groups which we index using the Artin upper numbering system. The slopes of D_p are the indices c of the non-trivial subquotients $Q_p^c = D_p^c / D_p^{>c}$. The group $Q_p^0 = D_p/I_p$, which corresponds to unramified extensions, is cyclic; the group Q_p^1, which corresponds to tame ramification, is cyclic of order prime to p; the groups Q_p^c for $c > 1$ correspond to wild ramification and are elementary abelian p-groups.

We define the slope content of K_p to be a set, allowing multiplicities, of ordered pairs (c_i, p_i), having the following property. The p_i are prime and for all slopes c,

$$\prod_{c_i=c} p_i = |Q_p^c|.$$

Thus our terminology "slope content" is in the spirit of "Jordan-Hölder content".

The primes p_i which appear here are 2, 3, and 5. Exploiting this fact we merely print the c_i in decreasing order using the following conventions. The prime $p_i = 5$ occurs only for the unique F_5 field, and we print the corresponding c_i in boldface. When $p_i = p$, we print c_i in plain type. Otherwise, we print c_i in italics. There are in fact only the following possibilities for the c_i, and we use abbreviations as follows:

At 2			At 3		
		0			*0*
		0			0
		1			*1*
4/3, 4/3	=	[4/3]			**1**
		2	5/4, 5/4	=	[5/4]
8/3, 8/3	=	[8/3]	3/2	=	t
		3			2
7/2	=	s	9/4, 9/4	=	[9/4]
		4	5/2	=	f

Here whenever the slopes 4/3, 8/3, 5/4, or 9/4 appear, they appear with multiplicity two, allowing our abbreviations. For example, an entry [8/3]*10* indicates in particular that $|D_2| = 24$ and $|I_2| = 12$; in fact here one has $D_2 \cong S_4$ and $I_2 \cong A_4$.

Galois root discriminants. An irreducible linear representation

$$\rho\colon D_p \to \mathrm{Aut}(V)$$

of D_p has a slope $s(\rho)$, namely the smallest $c \in [0, \infty)$ with $D_p^{>c}$ in the kernel of ρ. The Artin exponent $c(\rho)$ of ρ is then the slope $s(\rho)$ times the degree $\dim(V)$. Artin exponents of arbitrary representations are then defined by additivity. If ρ is induced from a permutation representation $r\colon D_p \to S_X$, then the discriminant of the p-adic algebra K_p corresponding to X is $p^{c(\rho)}$.

In the slot grd, we give the Galois root discriminant $\delta = 2^\alpha 3^\beta$ of K to two decimal places. Here α is calculated as the mean slope of D_2 acting on itself by left translations; similarly, β as the mean slope of D_3 acting on itself by left translations. Thus, for example, suppose in SC_2 we have printed [8/3]*10*. Of the twenty-four slopes, three-quarters are 8/3, one-sixth are 1, and one-twelfth are 0; this yields the average $\alpha = 13/6 \approx 2.17$.

Fields with the same Galois group and resolvent are sorted by increasing Galois root discriminant. Fields with small root discriminant are interesting for a number of reasons; for examples constructed from the fields here, see [**R1**].

Class numbers. In the slot h we have printed the class number of K when different from 1.

A defining equation. The next block gives the coefficients a_i of a polynomial

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \in \mathbb{Z}[x]$$

with $K = \mathbb{Q}[x]/f(x)$ the field under consideration. Here, for almost-canonicity, we have chosen f so that the sum $T_2(f)$ is minimal. If α_i are the complex roots of f,

$$T_2(f) := \sum_{i=1}^{n} |\alpha_i|^2.$$

2-adic and 3-adic information. The next two blocks of data for K give information about the p-adic algebra K_p, for $p = 2$ and then $p = 3$. We will describe these two blocks in parallel. The algebra K_p has the form

$$K_p = K_p^1 \times K_p^2 \times \cdots ,$$

each of the factors being fields, say of weakly decreasing degree. There is one line of data for each field different from $\mathbb{Q}_p$. Thus, here there can be at most two lines of data. Note that even on Table 6, the *a priori* possibility that K_p is the product of three quadratic fields in fact does not occur.

The entry f^e gives the inertial degree f and the ramification index e; thus the degree of the field is the product fe. The entry c_w gives the Artin exponent c and the root number w of the local field K_p^j. The entry d gives the discriminant class of K_p^j. Here the natural map $\{-6, -3, -2, -1, 1, 2, 3, 6\} \to \mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2}$ is conveniently bijective; we thus use these labels even in the current local context. On the other hand, the natural map $\{-6, -3, -2, -1, 1, 2, 3, 6\} \to \mathbb{Q}_3^\times/\mathbb{Q}_3^{\times 2}$ is surjective with kernel $\{1, -2\}$. Here we use the labels $\{-3, -1, 1, 3\}$ — one needs to always remember that it is -2 and not 2 that is trivial.

If there is no proper quadratic subfield of K_p^j, the entry s is left blank. If there is exactly one proper quadratic subfield, the entry s gives its discriminant class. The only other possibility is that there are three quadratic subfields of K_p^j; then $K_p^j = K_p^1$ is quartic with Galois group V. There are seven V quartics over $\mathbb{Q}_2$; but it happens that none of them appear as a K_2^1 for a field K on either of our tables. There is one V quartic over $\mathbb{Q}_3$; this appears as K_3^1 on our tables just for three D_4 quartics; in this case, we print V in the column s.

The paper [**R2**] tabulates all 2-adic and 3-adic fields of degree ≤ 6. The data (f^e, c_w, d, s) often suffice to determine K_p^j up to isomorphism. In degrees ≤ 5 there can be otherwise only a 2-fold or a 3-fold ambiguity. In these cases, we add an identifying label from $\{x, y\}$ or from $\{a, b, c\}$, according to the scheme in [**R2**].

The local information just described in fact determines most of the information in the first block of data. Namely the Artin exponent c_p decomposes additively while the root number w_p and the discriminant class d, considered in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$, decompose multiplicatively. For example, for the quartic S_4 field with discriminant $-^2 2^5 3^5$, and thus discriminant class $d = 6$, the decompositions at $p = 2$ are

$$\begin{aligned} c: &\quad 5 = 2 + 3 \\ w: &\quad i = i \cdot 1 \\ d: &\quad 6 = 3 \cdot 2. \end{aligned}$$

Note that for $\mathbb{Q}_p$ the relevant invariants are trivial: $c = 0$, $w = 1$, and $d = 1$.

This local information also determines the slope content in a less direct way. For example, suppose $n = 4$ and K_2 is a field. If $c = 9$ or 10, then the slope content is s32. If $c = 11$, then the slope content is 43, 430, or 432, according to whether the product discriminant class sd is 1, -3, or -1.

Frobenius data. The last block of data gives the Frobenius elements Fr_p for $p = 5, 7, \dots, 47$, as partitions of the degree n. To save space, we do not print 1's: thus the partition $4 = 2+1+1$ gets printed just as 2. This information is essential for identifying fields involved in the 2-adic or 3-adic representations associated to a motive. For given a motive, one typically has direct access only to Frobenius data, not to discriminantal data, and certainly not to defining equations.

1.3. Table for degrees ≤ 5. Table 1 presents the complete table of cubics, quartics, and quintics with discriminant $-^j 2^a 3^b$, excluding the V, C_4, and D_4 quartics as has been explained. As emphasized in the introduction, this list of fields is not new. The list of nine cubics is given in [**MS**], where these nine fields are studied in considerable detail. All but one of the cubics, quartics, and quintics appear on the large lists available in Pari format by anonymous ftp from [**B**]. The one exception is the highest discriminant quintic, which is covered by [**SPD**]. On the other hand, these large lists do not provide all the information that Table 1 does.

Among several things to note in Table 1, we'll just mention one. There are eight S_3 cubics and one A_3 cubic. For all but one of these cubics F, the 3-decomposition group $\mathrm{Gal}(F_3)$ coincides with global Galois group $\mathrm{Gal}(F)$. The exception is the unique cubic with discriminant class -2, for which the 3-local-to-global inclusion is $A_3 \subset S_3$. Now up to conjugation there are just two subgroups of S_4 surjecting down to S_3 under the resolvent map, namely S_4 itself and $S_3 \subset S_4$. Similarly, there are just two subgroups of S_4 surjecting to A_3, namely A_4 and $A_3 \subset A_4$. Now neither S_4 nor A_4 is of the form 3-group by $3'$-cyclic by cyclic; so they are not quotients of $\mathrm{Gal}(\overline{\mathbb{Q}}_3/\mathbb{Q}_3)$. In conclusion, the 3-completion of all of our A_4 or S_4 quartics K has the form $\mathbb{Q}_3 \times F_3$ where F_3 is the completion of the resolvent cubic F. This explains the repetitiveness of the 3-local block of columns.

TABLE 1. Low degree fields (Galois group $\neq V_4$, C_4, or D_4)

Discriminant class: -6

G	∞	2	**3**	a_1	a_2	a_3	Over $p=2$					Over $p=3$					5	7	11	13	17	19	23
grd	h	sC_2	sC_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_2	1_k	3_-	1_k	0	6		1^2	3_-	-6			1^2	1_k	3			+	+	+	−	−	−	−
4.90	2	3	*1*														+	+		−	−	−	−
S_3	1_k	3_-	3_k	0	3	-2	1^2	3_-	-6	-6		1^3	3_k	3			3	3	3	2	2	2	2
10.19		3	t*1*														e	3		2	2	2	2
$\widetilde{S_4}$	1_k	11_-	3_k	0	0	8	1^4	11_-	-6	-6x		1^3	3_k	3			3	3	3	4	4	2	2
24.24	4	43	t*1*	6													22	3		2	2	4	4
$\widetilde{S_4}$	1_k	9_-	3_k	0	0	4	1^4	9_-	-6	-1		1^3	3_k	3			3	3	3	4	4	4	4
24.24		s32	t*1*	-3													22	3		4	4	2	2
$\widetilde{S_4}$	1_k	11_-	3_k	0	-12	16	1^4	11_-	-6	6x		1^3	3_k	3			3	3	3	2	2	4	4
28.82		432	t*1*	12													22	3		4	4	4	4
S_5	1_k	11_+	5_i	0	-2	-4	1^4	11_+	-6	-6x		1^3	5_i	-3		c	5	5	3	32	32	32	32
50.41		43	f*10*	-9	-4							2	0_+	-1			3	5		4	4	4	4

Discriminant class: -3

G	∞	2	**3**	a_1	a_2	a_3	Over $p=2$					Over $p=3$					5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_2	1_k	0_+	1_i	0	3		2	0_+	-3			1^2	1_i	-3			−	+	−	+	−	+	−
1.73		0	*1*														−	+		+	−	+	−
S_3	1_k	2_+	3_i	0	0	-2	1^3	2_+	-3			1^3	3_i	-3			2	3	2	3	2	3	2
5.72		*1*0	t*1*														2	e		3	2	e	2
$\widetilde{S_4}$	1_k	8_+	3_i	0	0	4	1^4	8_+	-3			1^3	3_i	-3			4	3	4	3	4	3	4
16.18		[8/3]*1*0	t*1*	-6													2	22		3	2	22	4
S_3	1_k	0_+	5_i	0	0	-3	2	0_+	-3	-3		1^3	5_i	-3		a	2	3	2	3	2	3	2
7.49		0	f*1*														2	3		3	2	3	2
$\widetilde{S_4}$	1_k	6_+	5_i	2	-3	2	2^2	6_+	-3	-3x		1^3	5_i	-3		a	4	3	2	3	4	3	4
21.20	2	30	f*1*	7													2	3		3	4	3	2
$\widetilde{S_4}$	1_k	6_+	5_i	2	0	4	1^4	6_+	-3	-1		1^3	5_i	-3		a	4	3	4	3	2	3	2
21.20		220	f*1*	2													4	3		3	4	3	4
$\widetilde{S_4}$	1_k	8_+	5_i	0	-6	4	1^4	8_+	-3	3		1^3	5_i	-3		a	2	3	4	3	4	3	4
29.98		320	f*1*	6													4	3		3	2	3	4
S_3	1_k	2_+	5_i	0	0	-12	1^3	2_+	-3			1^3	5_i	-3		b	2	3	2	e	2	e	2
11.90		*1*0	f*1*														2	3		3	2	3	2
$\widetilde{S_4}$	1_k	8_+	5_i	0	-6	4	1^4	8_+	-3			1^3	5_i	-3		b	2	3	4	22	2	22	4
33.65		[8/3]*1*0	f*1*	15													4	3		3	4	3	2
S_3	1_k	2_+	5_i	0	0	-6	1^3	2_+	-3			1^3	5_i	-3		c	2	e	2	3	2	3	2
11.90		*1*0	f*1*														2	3		e	2	3	2
$\widetilde{S_4}$	1_k	4_+	5_i	2	0	6	1^4	4_+	-3			1^3	5_i	-3		c	4	22	4	3	4	3	4
16.82		[4/3]*1*0	f*1*	3													2	3		22	2	3	2

Discriminant class: -2

G	∞	**2**	3	a_1	a_2	a_3	Over $p=2$					Over $p=3$					5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_2	1_k	3_i	0_+	0	2		1^2	3_i	-2								−	−	+	−	+	+	−
2.83		3															−	−		−	+	+	−
S_3	1_k	3_i	4_+	0	-3	-10	1^2	3_i	-2	-2		1^3	4_+	1		b	2	2	3	2	3	3	2
12.24	3	3	2														2	2		2	3	3	2
$\widetilde{S_4}$	1_k	9_i	4_+	0	-6	8	1^4	9_i	-2	3		1^3	4_+	1		b	4	2	3	4	3	3	4
29.11		s32	2	6													2	2		4	3	3	4
$\widetilde{S_4}$	1_k	11_i	4_+	0	0	8	1^4	11_i	-2	6		1^3	4_+	1		b	4	4	3	2	3	3	2
29.11		430	2	-6													4	2		4	3	3	2
$\widetilde{S_4}$	1_k	11_i	4_+	0	-12	8	1^4	11_i	-2	2x		1^3	4_+	1		b	2	4	3	4	3	3	4
34.61	2	432	2	18													4	2		2	3	3	4

Discriminant class: -1

G	∞	**2**	3	a_1	a_2	a_3	Over $p=2$					Over $p=3$					5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_2	1_k	2_i	0_+	0	1		1^2	2_i	-1			2	0_+	-1			+	−	−	+	+	−	−
2.00		2	*0*														+	−		+	+	−	−
S_3	1_k	2_i	4_+	0	-3	-4	1^2	2_i	-1	-1		1^3	4_+	-1			3	2	2	3	3	2	2
8.65		2	2*0*														3	2		3	e	2	2
$\widetilde{S_4}$	1_k	6_i	4_+	2	3	6	2^2	6_i	-1	-3		1^3	4_+	-1			3	4	4	3	3	4	4
17.31		320	2*0*	3													3	2		3	22	4	2
$\widetilde{S_4}$	1_k	10_i	4_+	0	0	16	1^4	10_i	-1	-6		1^3	4_+	-1			3	2	4	3	3	2	4
29.11	2	s32	2*0*	-24													3	4		3	22	4	4
$\widetilde{S_4}$	1_k	10_i	4_+	0	-6	8	1^4	10_i	-1	2		1^3	4_+	-1			3	4	2	3	3	4	2
29.11	2	s32	2*0*	15													3	4		3	22	2	4

Discriminant class: 1

G	∞	2	3	a_1	a_2	a_3	Over $p=2$					Over $p=3$					5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_3	0_+	0_+	4_+	0	-3	1	3	0_+	1			1^3	4_+	1		a	3	3	3	3	e	e	3
4.33		0	2														3	3		e	3	3	3
A_4	2_-	6_-	4_+	2	6	4	1^4	6_-	1			1^3	4_+	1		a	3	3	3	3	22	22	3
12.24		22*0*	2	2													3	3		22	3	3	3

Discriminant class: 2

G	∞	2	3	a_1	a_2	a_3	Over $p=2$					Over $p=3$					5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_2	0_+	3_+	0_+	0	-2		1^2	3_+	2			2	0_+	-1			−	+	−	−	+	−	+
2.83		3	*1*														−	+		−	+	−	+
F_5	2_-	11_-	4_+	-2	-2	8	1^4	11_-	2	2x		1^5	4_+	-1			4	5	4	4	5	4	5
16.20		43	**1***00*	-1	-10												4	22		4	22	4	22
S_5	2_-	9_-	6_+	-1	-2	6	1^4	9_-	2	-1		1^3	5_i	-3		c	4	5	4	32	5	4	5
50.41		s32	f*10*	-6	6							1^2	1_k	3			4	3		32	5	32	5
S_5	2_-	11_-	6_+	-1	4	-12	1^4	11_-	2	-2x		1^3	5_i	-3		b	32	3	32	4	5	32	5
59.95		432	f*10*	12	-12							1^2	1_k	3			4	3		32	3	32	22

Discriminant class: 3

G	∞	**2**	**3**	a_1	a_2	a_3	Over $p=2$					Over $p=3$					5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_2	0_+	2_i	1_k	0	-3		1^2	2_i	3			1^2	1_k	3			−	−	+	+	−	−	+
3.46		2	*1*														−	−		+	−	−	+
S_5	2_-	10_i	5_i	-2	2	4	1^4	10_i	3	6		1^3	5_i	-3		a	32	4	5	3	32	4	5
50.41		s32	f*10*	-5	2							2	0_+	-1			4	32		5	4	32	3

Discriminant class: 6

G	∞	**2**	**3**	a_1	a_2	a_3	Over $p=2$					Over $p=3$					5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_2	0_+	3_k	1_i	0	-6		1^2	3_k	6			1^2	1_i	-3			+	−	−	−	−	+	+
4.90		3	*1*														+	−		−	−	+	+
S_3	0_+	3_k	5_i	0	-9	-6	1^2	3_k	6	6		1^3	5_i	-3		b	3	2	2	2	2	3	3
21.20		3	f*1*														3	2		2	2	e	3
S_4	2_-	5_i	5_i	0	3	2	1^2	2_i	3			1^3	5_i	-3		b	3	4	4	4	4	3	3
29.98		32	f*1*	6			1^2	3_+	2								3	2		2	4	22	3
S_4	2_-	9_i	5_i	0	12	16	1^4	9_i	6	3		1^3	5_i	-3		b	3	2	2	4	2	3	3
50.41	4	s32	f*1*	24													3	4		2	4	22	3
S_4	2_-	11_i	5_i	0	12	16	1^4	11_i	6	-2		1^3	5_i	-3		b	3	2	4	2	2	3	3
50.41	2	430	f*1*	6													3	2		2	4	22	3
$\widetilde{S_4}$	0_+	11_k	5_i	0	-24	32	1^4	11_k	6	-2		1^3	5_i	-3		b	3	4	2	4	4	3	3
50.41		430	f*1*	24													3	2		2	2	22	3
S_4	2_-	9_i	5_i	0	12	4	1^4	9_i	6	3		1^3	5_i	-3		b	3	4	4	2	4	3	3
50.41	4	s32	f*1*	69													3	4		2	2	e	3
S_4	2_-	11_i	5_i	0	12	8	1^4	11_i	6	-6x		1^3	5_i	-3		b	3	2	4	4	2	3	3
59.95	2	432	f*1*	42													3	4		2	2	22	3
S_4	2_-	11_i	5_i	0	12	16	1^4	11_i	6	-6y		1^3	5_i	-3		b	3	4	2	2	4	3	3
59.95	2	432	f*1*	60													3	4		2	4	22	3
S_5	2_-	11_i	5_i	-2	4	0	1^4	11_i	6	-6y		1^3	4_+	1		c	5	4	4	32	4	5	3
41.57		432	2*1*	-6	12							1^2	1_i	-3			5	4		32	32	3	5

2. Targeted searches

Let n and D be positive integers. There is a standard method by which one can make systematic searches for all primitive number fields of degree n and absolute discriminant $\leq D$. Here we quickly review this method. Then we explain, in more detail, how one can modify it to make much quicker searches for primitive degree n number fields with absolute discriminant exactly D. It is a question of supplementing the usual archimedean bounds on coefficients with p-adic bounds for each prime p dividing D. Throughout, we illustrate the generalities with our particular case $n = 6$ and D of the form $2^a 3^b$.

Let K a number field of the type sought: primitive, degree n, and with absolute discriminant D. Let $\mathcal{O}_K$ be its ring of integers and I the product of all prime ideals in $\mathcal{O}_K$ over primes p dividing D. For $\eta \in I$ one has its characteristic polynomial

$$f_\eta(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n \in \mathbb{Z}[x].$$

In §2.1, we state a version of Hunter's theorem adapted to this context. It guarantees the existence of one $\eta \in I - \mathbb{Z}$ with the corresponding coefficients a_i satisfying a certain quite complicated system of inequalities. In §2.2, we introduce integers $v_{p,i}$. They are the largest integers such that every $\eta \in I$ satisfies the congruence $p^{v_{p,i}} | a_i$. Details of our computer search are given in §§2.3–2.4, and possible improvements in §2.5.

2.1. Archimedean bounds. The traditional statement of Hunter's theorem involves the full ring of integers $\mathcal{O}_K$. For targeted searches, we use the completely analogous statement for our search ideal I.

Theorem 2.1. *Let K be a degree n number field with absolute discriminant D. Let l be the least positive integer contained in I and let m be the order of $\mathcal{O}_K/I$. Finally, let γ_n be Hermite's constant for n-dimensional lattices. Then there exists an element $\eta \in I - \mathbb{Z}$ such that*

$$T_2(f_\eta) \leq \frac{\mathrm{Tr}(\eta)^2}{n} + \gamma_{n-1} \left(\frac{m^2 D}{l^2 n} \right)^{1/(n-1)}$$

and $0 \leq \mathrm{Tr}(\eta) \leq [nl/2]$.

For $I = \mathcal{O}_K$, this is the usual statement of Hunter's theorem as it appears, for example, in [**C**]. The new factor m^2/l^2 in the discriminant term gives in practice much weaker archimedean bounds. The ultrametric bounds in §2.2 more than make up for this loss. As in the case when $I = \mathcal{O}_K$, search times are dependent essentially on only the discriminant term. In our searches, $n = l = 6$ and Hermite's constant $\gamma_5 = \sqrt[5]{8}$, so the discriminant term is $(m^2 D/27)^{1/5}$.

The transition from this bound on $T_2(f_\eta)$ to explicit bounds on the coefficients of f_η is very subtle and a point of continuing research. In our computer program we use the methods described in [**P**, §3].

Remark 2.2. We are interested in both primitive fields and imprimitive fields, yet the version we have stated of Hunter's theorem is not suitable for classifying imprimitive fields K. The problem is that the element η, whose existence is being asserted, might lie in a proper subfield of K. There is a version of Hunter's theorem which applies to imprimitive fields due to Martinet [**Mar**]. However our search for primitive fields in fact found all imprimitive fields as well, as we will prove by sextic twinning and class field theory in §3.

2.2. Ultrametric bounds. Fix p dividing the targeted absolute discriminant D; thus, $p = 2$ or $p = 3$ for the computer search conducted for this paper. Let c be the Artin exponent at p, so that p^c exactly divides D. The considerations of this section are purely p-adic; we consider p-adic algebras K_p of degree n and discriminant p^c.

The target p^c is best thought of as composed of several smaller targets. Accordingly, one aims at these smaller targets separately as follows. Let $\mathbb{Q}_p^{\mathrm{un}}$ be a maximal unramified extension of $\mathbb{Q}_p$. Consider the base-changed algebra $K_p^{\mathrm{un}} = K_p \otimes \mathbb{Q}_p^{\mathrm{un}}$. It has a canonical decomposition into fields:

$$K_p^{\mathrm{un}} = K_p^{\mathrm{un},1} \times \cdots \times K_p^{\mathrm{un},w}.$$

Here we let e_j be the degree of $K_p^{\mathrm{un},j}$ over $\mathbb{Q}_p^{\mathrm{un}}$ and c_j its Artin exponent. Thus $\sum e_j = n$ and $\sum c_j = c$. We define a *ramification structure* to be a set of pairs which can arise in this way, allowing multiplicities of course:

$$r = \{(e_1, c_1), \ldots, (e_w, c_w)\}.$$

These ramification structures are the smaller targets.

Let K_p be a degree n algebra with ramification structure r as above, say written with the e_i weakly decreasing. Let I_p be the product of the maximal ideals in $\mathcal{O}_{K_p}$; this is now our local search ideal. Let $\eta \in I_p$ with characteristic polynomial $f_\eta(x) = x^n + a_1 x^{n-1} + \cdots + a_n$. From the e_i's alone, one knows that $p^{u_i} \mid a_i$ where

$$u_i = \min\left\{ u : \sum_1^u e_j \geq i \right\}.$$

These congruences are usually phrased in terms of Newton polygons: u_i is the smallest integer such that (i, u_i) is on or above the Newton polygon of K_p.

Recall that if $p \nmid e_j$ the field $K_p^{\mathrm{un},j}$ is tame, and very simply $c_j = e_j - 1$. If $p | e_j$ then the field $K_p^{\mathrm{un},j}$ is wild; its Artin exponent c_j is one of finitely many possibilities, the smallest being e_j and the largest being $\mathrm{ord}_p(e_j)e_j + e_j - 1$.

Define v_i to be the largest integer such that $p^{v_i} | a_i$ for all η in the search ideal I_p of any algebra K_p with the given ramification structure r. When each c_j is minimal for e_j one has very simply $v_i = u_i$. When some c_j are non-minimal one may have $v_i > u_i$ for some i. We call the v_i *Newton-Ore exponents* because of the connection with both Newton polygons and Ore's formulas for discriminants of Eisenstein polynomials [**Or**].

For $r = \{(n, c)\}$, corresponding to totally ramified fields, the Newton-Ore exponents have been described in many places; the discussion in [**R2**] gives examples particularly relevant for this paper. For $r = \{(e_1, c_1), \ldots, (e_w, c_w)\}$, the numbers v_i are computed by examining the coefficients of $f_1 \cdots f_w$; here f_j has degree e_j and is the general polynomial with coefficients satisfying the corresponding Newton-Ore congruences; a particular case is examined closely in §2.5 below.

We have calculated the Newton-Ore exponents for all possible sextic ramification structures $\{(e_1, c_1), \ldots, (e_w, c_w)\}$, treating the cases $p = 2$ and $p = 3$ separately. Tables 2 and 3 record some of the results. Here is an example, which illustrates how to read the tables. Take $p = 2$ and r of the form $\{(6, c)\}$, corresponding to totally ramified sextics. The relevant column is 6_c; in general, the tables present ramification structures as subscripted partitions of 6. The possible values for the Artin exponent c are 6, 8, 10, and 11. If c is 6, one has only the Newton congruences $2|a_1$, a_2, a_3, a_4, a_5, a_6. Table 2 shows the exponents v_i in sequence, 111111. As c increases one gets stronger congruences until for $c = 11$ one has $2|a_2$, a_4, a_6 and $4|a_1$, a_3, a_5, which Table 2 records as 212121.

2.3. Combining cases. There are 37 sextic ramification structures at $p = 2$ and 24 sextic ramification structures at 3. So there are $37 \cdot 24$ searches to be done, each with its own archimedean and ultrametric bounds. However many of our

Table 2. Newton-Ore exponents for $p = 2$

$4_c 2_2$				$4_c 2_3$				6_c			
c,2	Cong	α_1	α	c,3	Cong	α_1	α	c	Cong	α_1	α
11,2	113122	17	24	11,3	213132	18	24	11	212121	13	17
10,2	"	16	22	10,3	"	17	22	10	112121	12	16
9,2	112122	15W	21	9,3	"	16	20	8	111121	10	13
8,2	"	14	19	8,3	212132	$15w_1$	19	6	111111	8	10
6,2	"	12	15	6,3	112132	$13w_2$	16				
4,2	111122	10x	12	4,3	111122	11X	14				

Table 3. Newton-Ore exponents for $p = 3$

$3_c 111$				$3_c 3_d$				6_c			
c	Cong	β_1	β	c,d	Cong	β_1	β	c	Cong	β_1	β
5	121234	13y	13	5,5	221332	14	15	11	221221	13	16
4	"	12	11	4,5	121232	13Y	15	10	121221	12	15
3	111234	11z	10	4,4	"	12	13	9	111221	11	14
				3,5	111222	12Z	15	7	111121	9	11
				3,4	"	11	13	6	111111	8	10
				3,3	"	10	11				

smaller searches are rendered unnecessary by our larger searches, as we explain now.

Fix a ramification structure

$$r = \{(e_1, c_1), \ldots, (e_w, c_w)\}$$

with Artin exponent $c = \sum c_i$. Let

$$v = (v_1, v_2, \ldots, v_6)$$

be the associated sequence of Newton-Ore exponents; so $v_6 = w$. The ramification structure r contributes p^{2v_6+c} to the critical quantity $m^2 D$ in Hunter's theorem. The exponent $2v_6 + c$ has been printed in Tables 2 and 3, under the heading α_1 or β_1.

Consider another ramification structure $R = \{(E_1, C_1), \ldots, (E_W, C_W)\}$, inducing the Artin exponent C and the Newton-Ore exponents $(V_1, \ldots, V_6)$. We say that R *subsumes* r iff

$$V_i \le v_i \quad \text{for all } i = 1, \ldots, 6$$
$$2V_6 + C \ge 2v_6 + c.$$

We write $r \preceq R$ to indicate this relationship. The point is that the search for all R_2-R_3 fields finds also all r_2-r_3 fields if $r_2 \preceq R_2$ and $r_3 \preceq R_3$.

For example, on the above table any ramification pattern r_p indicated by quotes is subsumed by the ramification pattern R_p above it, to which the quotes refer. Also, one has the five indicated relations $w_1 \preceq W$, $w_2 \preceq W$, $\ldots$, $z \preceq Z$. In particular, the entire column $3_c 111$ for $p = 3$ is subsumed by other entries. Among the 37 ramification patterns at 2, the maximal ones are exactly the 8 patterns maximal on Table 2; similarly, among the 24 ramification patterns at 3, the maximal ones are exactly the 8 patterns which are maximal on Table 3. Thus, we in fact carry out $64 = 8 \cdot 8$ separate searches, rather than $888 = 37 \cdot 24$ searches.

2.4. The computer searches. Each of the 64 searches was carried out in two stages. The program for the first stage was written in C, utilizing the Pari programming library, and was implemented on several Sun and HP workstations. The program was designed to find all sextic monic polynomials in $\mathbb{Z}[x]$ with coefficients a_i satisfying the archimedean, 2-adic, and 3-adic inequalities discussed above. By far, this was the more time-consuming stage.

The second stage checked for irreducibility on early polynomials, as defined below. It then eliminated duplicate fields, and generated most of the information found in Tables 1 and 6.

The first stage ran each polynomial f through the following screening procedure.

(1) Compute the absolute polynomial discriminant of f, and divide out all factors of 2 and 3. If the result is 1, output f to stage 2. If the result is a square different from 1, pass f on to step 2.
(2) Pass f to step 3 iff $T_2(f)$ is below the Hunter bound.
(3) Pass f to step 4 iff f is irreducible.
(4) Output f to stage 2 iff the absolute field discriminant $|D_{K(f)}|$ is of the form $2^a 3^b$.

The order of the tests above is important, typical times for Test i being very roughly as given in the second column of Table 4; these times are averages based on 1000 sample polynomials. The third and fourth columns give some data for 2 of

TABLE 4. Timing data

	Time per polynomial	6_{11} at 2 6_{10} at 3	6_{11} at 2 $3_5 3_5$ at 3
Test 1:	0.0013 Seconds	1017101	1301618
Test 2:	0.055 Seconds	4014	4779
Test 3:	0.104 Seconds	1438	2194
Test 4:	0.128 Seconds	1383	2064
Late Fields Found		140	203
Early Fields Found		384	402
Distinct Fields Found		229	215
Distinct Fields on Target		1	11

our 64 searches. The first line gives the number of polynomials inspected, i.e. the number to which we applied Test 1. The second line gives the number which passed Test 1 to which we then applied Test 2, and so on. *Late fields found* indicates the number of polynomials output by Test 4, whereas *early fields found* indicates the number of irreducible polynomials output by Test 1. *Distinct fields found* represents the result of combining early and late fields and removing duplicates, while *distinct fields on target* represents those which match the targeted ramification structure.

Note that Test 1 is not available when one is looking simply for all fields of degree n with absolute discriminant less than a given bound D; so here we have an important time savings inherent in targeted searches. As the numbers indicate, relatively few polynomials need further testing.

The entries α and β in Tables 2 and 3 are calculated by $\alpha = 2\alpha_1 - \sum v_{2,i}$ and $\beta = 2\beta_1 - \sum v_{3,i}$. The time required for a given case is roughly proportional to the number of polynomials inspected, which is in turn proportional to $2^\alpha 3^\beta$. Our searches for the longest cases, namely $4_{11}2_c$ at 2 versus 6_{11} at 3, for $c = 2, 3$, inspected approximately half a billion polynomials each.

Remark 2.3. The following brief discussion of the large drop $229 \to 1$ on Table 4 throws some further light on the nature of our search. To search for fields with ramification structure 6_{11} at 2 and 6_{10} at 3, we imposed the congruences $4|a_1$, a_3, a_5, $2|a_2$, a_4, a_6, $3|a_1$, a_3, a_6, and $9|a_2$, a_4, a_5, in accordance with Tables 2 and 3. Normalize Haar measure so that the corresponding region R in $\mathbb{Z}_2^6 \times \mathbb{Z}_3^6$ has total mass one. Let $T \subset R$ be the subregion of R consisting of polynomials with exactly the targeted ramification structure. The region T contains T_0 which is defined by the additional conditions $4 \nmid a_6$ ("Eisenstein at 2"), $9 \nmid a_6$ ("Eisenstein at 3"), and $9 \nmid a_1$ (forcing $c_3 = 10$ rather than $c_3 = 11$). The mass of T_0 is $(1/2)(2/3)(2/3) = 2/9$.

In searching all of R, rather than just the target T, we are over-searching by a factor of less than 4.5, reasonable given the circumstances. In the region T, in fact T_0, we found exactly the one targeted field, listed as the second S_6 field on our $d = 2$ subtable. In retrospect one sees that our search over R found more than half of the 398 sextic fields with discriminant $-^j 2^a 3^b$. This fact serves as a reminder that it is relatively easy to produce *many* sextic fields with discriminant $-^j 2^a 3^b$; the deep assertion in this paper is that we have found *all* such sextic fields.

2.5. Improvements. Our search was carried out using easy-to-program bounds, but by no means optimal bounds. For larger n and/or larger targeted absolute discriminant D it would be important to use better bounds.

Better archimedean bounds. This has been explored by several papers, e.g. [**SPD**]. The best results target the different possible ramification patterns $-^j$ separately. For $j = 0$, i.e. the totally real case, this is a very substantial savings. As j increases, the savings becomes more modest. Our archimedean bounds were actually moderately close to optimal since roughly one-third of the polynomials f we inspected indeed had $T_2(f)$ under the search bounds.

Better ultrametric bounds. When the target ramification structures are totally ramified, our use of coefficient-by-coefficient congruences only is reasonably close to optimal, as illustrated by Remark 2.3. However in general, it would be important to take advantage of other more complicated congruences as we indicate by the following example.

Take $p = 2$ and target the ramification structure $4_{11}2_2$. The relevant product is then

$$\begin{aligned}(x^2 + 2ax + 2b)&(x^4 + 8Ax^3 + 4Bx^2 + 8Cx + 2D)\\ &= x^6 + 2(a + 4A)x^5 + 2(b + 2B + 8aA)x^4 + 8(aB + C + 2Ab)x^3\\ &\qquad + 2(D + 4bB + 8aC)x^2 + 4(aD + 4bC)x + 4(bD).\end{aligned}$$

We used only the obvious congruences $2^{v_i} \mid a_i$ with $(v_1, v_2, v_3, v_4, v_5, v_6) = (1, 1, 3, 1, 2, 2)$, as stated in Table 2. Thus we searched over a lattice with coordinates

$$\begin{aligned} b_1 &= a + 4A & b_2 &= b + 2B + 8aA \\ b_3 &= aB + C + 2Ab & b_4 &= D + 4bB + 8aC \\ b_5 &= aD + 4bC & b_6 &= bD. \end{aligned}$$

Among the congruences we did not exploit are the following.

$$b_5 \equiv b_1 b_4 \quad (4)$$
$$\text{If } b_4 \equiv 1 \quad (2) \text{ then } b_6 \equiv b_2 b_4 \quad (2)$$
$$\text{If } b_4 \equiv 0 \quad (2) \text{ then } b_6 \equiv b_2 b_4 \quad (4).$$

In fact, these are all the congruences modulo 4. Namely, as (a, b, A, B, C, D) runs over the 4096 elements of $(\mathbb{Z}/4)^6$, the vector $(b_1, b_2, b_3, b_4, b_5, b_6)$ runs over exactly the 384 elements of $(\mathbb{Z}/4)^6$ satisfying these congruences. So the incorporation of just these congruences into the program would give a time savings of roughly a factor of 10 for the cases involving the ramification structure $4_{11}2_2$ at 2.

3. Sextics

In §3.1, we give the complete breakdown of the 398 fields found by the computer search according to Galois group and discriminant class. In §3.2, we prove that the list of 282 old sextic fields is complete using the tables of smaller degree fields from §1.3 and sextic twinning. In §3.3, we tabulate the 116 new sextic fields. Finally in §3.4, we prove that the list of new solvable sextic fields is complete using class field theory.

3.1. Summarizing table. The sextic twinning operator t is treated in detail in [**R3**]. Here we review just enough for our purposes. If X is a six element set, one has canonically a second six-element set X^t with a natural bijection $X \to X^{tt}$. Correspondingly, if K is a sextic separable algebra over a field F then one has its twin algebra K^t. The Galois groups G of K and G^t of K^t are identical as quotient groups of $\mathrm{Gal}(\overline{F}/F)$; however the permutation representations X and X^t are not isomorphic, except in degenerate cases where the Galois group is very small. Given a defining polynomial of a sextic separable algebra, a defining polynomial for its twin algebra can be computed as a resolvent sextic for the subgroup $PGL_2(5) \subset S_6$.

If K is a field, there are two possibilities. First, its Galois group G may be $C_3^2.C_4$, $C_3^2.D_4$, A_6, or S_6. In this case, K^t is a field not isomorphic to K. These "new" fields thus come in twin pairs. Second, the Galois group G may be one of the remaining twelve transitive subgroups of S_6. In this case K^t is not a field, its Galois group G^t not being transitive. Thus these "old" fields are by definition constructible from fields of smaller degree.

Table 5 gives, for each of the 16 transitive subgroups G of S_6, and each of the eight possible discriminant classes $d \in \{-6, -3, -2, -1, 1, 2, 3, 6\}$, the number of sextic fields with Galois group G and discriminant class d found by the search. The top block summarizes Table 1. The format is similar to the next three blocks which we describe now. In the column headed by G, we list the 11 transitive subgroups of S_6 which are not in A_6. The column labeled by G_+ gives information about $G \cap A_6$. In five cases this subgroup is not transitive; then the entry is left blank. In six cases the subgroup is transitive; then it is given, with the unique exception of the even subgroup of $S_2 \wr^\epsilon S_3$. This subgroup is $S_2 \wr^+ A_3$ which has already appeared as the even subgroup of $S_2 \wr A_3$. Our notation for subgroups follows [**R3**] and, in particular, "$\wr$" indicates a wreath product.

TABLE 5. Summarizing table for degrees ≤ 6

$\lvert G_+\rvert$	$\lvert G\rvert$	$G_+ \subset G$	$G_+^t \subset G^t$	-6	-3	-2	-1	1	2	3	6	Tot
Some cubics, quartics, and quintics												
3	6	$A_3 \subset S_3$		1	4	1	1	**1**	0	0	1	**8**
12	24	$A_4 \subset S_4$		3	6	3	3	**1**	0	0	7	**22**
60	120	$A_5 \subset S_5$		1	0	0	0	**0**	2	1	1	**5**
Old sextics												
	6	C_6	A_3S_2	1	1	1	1		1	1	1	**7**
	6	$S_{3\text{gal}}$	S_3	1	4	1	1		0	0	1	**8**
	12	D_6	S_3S_2	7	4	7	7		8	8	7	**48**
	18	$C_3^2.C_2$	A_3S_3	1	4	1	1		0	0	1	**8**
	36	$C_3^2.V$	S_3S_3	1	1	4	1		5	5	5	**22**
12	24	$S_2 \wr^+ A_3 \subset S_2 \wr A_3$	$A_4 \subset A_4S_2$	1	1	1	1	**1**	1	1	1	**7**
	24	$\subset S_2 \wr^\epsilon S_3$	$\subset S_4$	3	6	3	3		0	0	7	**22**
24	48	$S_2 \wr^+ S_3 \subset S_2 \wr S_3$	$S_4^+ S_2 \subset S_4S_2$	19	16	19	19	**22**	22	22	15	**132**
60	120	$\mathrm{PSL}_2(5) \subset \mathrm{PGL}_2(5)$	$A_5 \subset S_5$	1	0	0	0	**0**	2	1	1	**5**
New solvable sextics												
36	72	$C_3^2.C_4 \subset C_3^2.D_4$	$C_3^2.C_4 \subset C_3^2.D_4$	10	10	4	18	**4**	8	0	0	**50**
New non-solvable sextics												
360	720	$A_6 \subset S_6$	$A_6 \subset S_6$	2	2	0	4	**8**	26	8	12	**54**

On the G row, we have also printed $(G_+^t \subset)G^t$. Here $G = G^t$ but G is acting on a six element set X, while G^t is acting on the six element set X^t. In the old cases, the action of G^t is intransitive, and the subscripts indicate the orbit structure, fixed points not being mentioned. For example, $D_6 = S_3S_2$ acts transitively on X and with orbit partition 321 on X^t.

Finally in the G row, the entries under $d \neq 1$ refer to G, while the entry under $d = 1$, when non-blank, refers to G_+. Thus the sum of the sixteen numbers in boldface corresponding to sextics is the number of sextic fields with discriminant $-^j 2^a 3^b$ found by the search, namely 398.

In the case of sextic fields, one has something of a coincidence: a transitive subgroup of S_6 is primitive iff it is non-solvable. Thus the computer search was guaranteed to find all the fields only in the cases $G = \mathrm{PSL}_2(5)$, $\mathrm{PGL}_2(5)$, A_6, and S_6.

3.2. Old fields. Here we use Table 1 and sextic twinning to prove that the numbers of old sextic fields presented in the previous section are correct, giving a total of 282 old sextic fields. This shows that the computer search, which was not guaranteed to find all old fields, in fact did. We then give a short table giving defining polynomials for five particularly interesting old sextic fields.

Consider first the transitive group $S_{3\text{gal}}$. Sextic $S_{3\text{gal}}$ fields are in bijection with cubic S_3 fields via twinning, i.e. via Galois closure in this case. Hence, the $S_{3\text{gal}}$-d entry on Table 5 must be just the S_3-d entry. The cases with $G^t = S_4$, A_4, S_5, and A_5 are similarly trivial.

Consider next the transitive group C_6. These fields are in bijection with ordered pairs of appropriate fields (L_3, L_2) by twinning, i.e. by tensor product in this case. So the C_6-d entry on Table 5 is the product of the A_3-1 entry and the S_2-d entry, i.e. just $1 \cdot 1 = 1$. The cases with $G^t = A_3S_3$ and A_4S_2 are similarly trivial.

Now consider the transitive group $G = C_3^2.V$ with twin group $G^t = S_3S_3$. By twinning $C_3^2.V$-d sextics are in bijection with ordered pairs (L_a, L_b) of S_3 cubics with discriminant classes satisfying say $d_a < d_b$ and $d_a d_b = d$. The simple "multiplication" table below lets one easily deduce the desired numbers. For example, there are five $C_3^2.V$ sextics of discriminant class 2, corresponding to the sum of the boldface table entries.

	d_b	-6	-3	-2	-1	6
d_a	#	1	4	1	1	1
-6	1		**4**	1	1	1
-3	4			4	4	4
-2	1				**1**	1
-1	1					1
-6	1					

Sextic fields with group $G = S_2 \wr^+ S_3$ are in bijection with quartic fields with group S_4; there are thus 22 of them, as indicated by Table 5. Similarly, consider sextic fields with group $G = S_2 \wr S_3$ and discriminant class d. Since $G^t = S_4S_2$, these fields are in bijection with quartic fields with group S_4 and discriminant class different from d. Thus, for example, there are $22 - 3 = 19$ sextic fields with group $S_2 \wr S_3$ and discriminant class $d = 6$. This finishes our completeness proof.

To save space, we do not tabulate all 282 old sextics here. Tables are currently available from [**J**]. We wish however to emphasize that these fields K are of some interest as well, not everything being immediately deducible from knowledge of the twin algebra K^t. For example, the integers arising as class numbers of $S_2 \wr S_3$ sextics are 1, 2, 3, 4, 6, 7, 8, 12, and 24. Since large class numbers are often of particular interest we list the five fields giving the four largest class numbers.

h	Cl	d	Irreducible Sextic	$\leftrightarrow$	Reducible Sextic
7	C_7	-2	$x^6 + 15x^4 + 48x^2 + 32$	$\leftrightarrow$	$(x^4 - 24x^2 + 32x + 24)(x^2 + 3)$
8	C_4C_2	-3	$x^6 + 18x^4 + 72x^2 + 48$	$\leftrightarrow$	$(x^4 - 24x^2 + 32x + 24)(x^2 + 2)$
8	C_4C_2	-6	$x^6 + 9x^4 + 18x^2 + 6$	$\leftrightarrow$	$(x^4 - 24x^2 + 32x + 24)(x^2 + 1)$
12	C_6C_2	-6	$x^6 - 6x^4 + 96$	$\leftrightarrow$	$(x^4 - 6x^2 + 8x + 6)(x^2 - 3)$
24	$C_{12}C_2$	-3	$x^6 + 6x^4 - 18x^2 + 12$	$\leftrightarrow$	$(x^4 + 8x - 6)(x^2 - 6)$

All these fields are totally imaginary. The quartic polynomial $x^4 - 24x^2 + 32x + 24$ is our preferred defining polynomial for the unique totally real S_4 quartic. In general the unique cubic subfield of an $S_2 \wr S_3$ sextic is the resolvent cubic of the associated-by-twinning quartic. So, the cubic subfield of the first three fields is totally real; thus these three fields are CM fields.

3.3. Table of new sextics. Table 6 has two lines for each of the twenty-eight D_4 fields and each of the four C_4 fields. The D_4 fields come in seven groups of four, each group being listed in an order F, F^t, F^T, and F^{tT}. Here one of the four D_4 quartics F is followed by its twin, its twist, and its twin-twist. The twin F^t is defined to be the unique quartic field which is distinct from F but has the same Galois group as a quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Twisting refers to replacing the permutation representation $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to D_4$ by the different permutation representation $\rho\chi$ where $\chi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to Z \subset D_4$ is a quadratic character with values in the center Z of D_4. The subfields of the four fields have the form $\mathbb{Q}(\sqrt{s})$, $\mathbb{Q}(\sqrt{s^t})$, $\mathbb{Q}(\sqrt{s})$, and $\mathbb{Q}(\sqrt{s^t})$ for s, s^t, ss^t distinct in $\{-6,-3,-2,-1,2,3,6\}$. These four fields have all been put on the $d = ss^t$ subtable. Similarly, all four C_4 quartics have been put on the $d = 1$ subtable.

The twin pair (F, F^t) is followed by twin pairs of $C_3^2.D_4$ sextics (K_a, K_a^t), (K_b, K_b^t), ... Here F is the resolvent quartic for K_a, K_b, ... while F^t is the resolvent quartic for K_a^t, K_b^t, ... Correspondingly, the twin pair (F^T, F^{tT}) is followed by twin pairs of $C_3^2.D_4$ sextics associated to it. Similarly, each C_4 quartic is followed by the twin pairs of $C_3^2.C_4$ sextics for which it is the resolvent. Next come the S_6 or A_6 sextics in twin pairs. Note that much of the information about K^t can be deduced from the corresponding information about K. We have printed the K^t information in accordance with our general policy that redundancy is acceptable if it improves clarity; here the tables make it clear how sextic twinning works on a practical level, both locally and globally.

The entries in the SC_p slots are slightly different from those in Table 1. For a pair of twin fields, we print the slope content as before for the first field. For its twin we print the Galois mean slopes to two decimal places for $p = 2$ and $p = 3$. Recall from §1.2 that these are rational numbers α and β satisfying $\mathrm{grd}(K) = 2^\alpha 3^\beta$.

Similarly, the labeling convention here is slightly different from the convention in Table 1. We label p-adic fields of degree ≤ 5 just as before, to distinguish among fields with the same invariants (f^e, c_w, d, s). Old sextic p-adic fields are typically not determined by their invariants (f^e, c_w, d, s). The worst case is sextic 3-adic fields with $(f^e, c_w, d, s) = (1^6, 11_k, -3, -3)$, where there are 15 different fields. These old fields are distinguished by the invariants—including labels—of their twin algebras; accordingly we do not label old sextic fields. There are no new sextic 2-adic fields; there are 12 new sextic 3-adic fields, which we label when necessary as before.

TABLE 6. New sextic fields

Discriminant class: -6

G	∞	2	**3**	a_1	a_2	a_3	Over $p=2$ f^e	c_w	d	s	L	Over $p=3$ f^e	c_w	d	s	L	5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_2	1_k	3_-	1_k	0	6		1^2	3_-	-6			1^2	1_k	3			+	+	+	−	−	−	−
4.90	2	3	*1*														+	+		−	−	−	−
$\widetilde{D_4}$	1_k	9_i	2_+	0	2	0	1^4	9_i	-2	3		2^2	2_+	1		V	4	4	e	2	22	22	2
11.65		s32	*10*	-2													4	4		2	22	22	2
D_4	2_-	10_i	1_i	0	-2	0	1^4	10_i	3	-2		1^2	1_i	-3			4	4	e	22	2	2	22
11.65		2.75	0.50	3								2	0_+	-1			4	4		22	2	2	22
$\widetilde{C_3^2.D_4}$	1_k	11_-	9_k	0	-6	-8	1^4	9_i	-2	3		1^6	9_k	3	3		42	42	33	2	6	222	32
44.62		s32	2t*10*	-18	24	16	1^2	2_i	3								42	42		32	6	6	32
$\widehat{C_3^2.D_4}$	3_i	13_-	7_i	0	0	8	1^4	10_i	3	-2		1^3	3_i	-3			42	42	3	222	32	2	6
44.62		2.75	1.72	18	24	24	1^2	3_i	-2			1^3	4_+	-1			42	42		6	32	32	6
$\widetilde{C_3^2.D_4}$	1_k	11_-	11_k	0	-18	24	1^4	9_i	-2	3		1^6	11_k	3	3		42	42	3	32	6	6	2
50.41		s32	f*10*	54	-144	96	1^2	2_i	3								42	42		32	6	222	32
$\widehat{C_3^2.D_4}$	3_i	13_-	5_i	2	-1	-4	1^4	10_i	3	-2		1^3	5_i	-3		c	42	42	33	6	32	32	222
50.41		2.75	1.83	-2	4	6	1^2	3_i	-2			2	0_+	-1			42	42		6	32	2	6
$\widetilde{C_3^2.D_4}$	1_k	11_-	11_k	0	0	0	1^4	9_i	-2	3		1^6	11_k	3	3		42	42	33	32	222	6	32
72.71		s32	f2*10*	-27	-36	-12	1^2	2_i	3								42	42		2	222	6	2
$\widehat{C_3^2.D_4}$	3_i	13_-	9_i	0	6	8	1^4	10_i	3	-2		1^3	4_+	-1			42	42	3	6	2	32	6
72.71		2.75	2.17	27	0	24	1^2	3_i	-2			1^3	5_i	-3		b	42	42		222	2	32	222
$\widetilde{C_3^2.D_4}$	1_k	11_-	11_k	0	-18	24	1^4	9_i	-2	3		1^6	11_k	3	3		42	42	3	32	6	6	32
72.71		s32	f2*10*	81	-216	132	1^2	2_i	3								42	42		32	6	6	32
$\widehat{C_3^2.D_4}$	3_i	13_-	9_i	0	-12	8	1^4	10_i	3	-2		1^3	5_i	-3		a	42	42	33	6	32	32	6
72.71		2.75	2.17	54	-72	24	1^2	3_i	-2			1^3	4_+	-1			42	42		6	32	32	6
D_4	1_k	9_k	2_-	0	-2	0	1^4	9_k	-2	3		1^2	1_k	3			4	4	22	2	22	22	2
11.65		s32	*1*	-2								1^2	1_k	3			4	4		2	22	22	2
D_4	2_-	10_k	1_k	0	2	0	1^4	10_k	3	-2		1^2	1_k	3			4	4	22	22	2	2	22
11.65		2.75	0.50	3													4	4		22	2	2	22
$C_3^2.D_4$	1_k	11_+	9_i	0	-6	-12	1^4	9_k	-2	3		1^6	9_i	3	3		42	42	22	32	222	6	32
44.62		s32	2t*1*	9	36	-12	1^2	2_i	3								42	42		32	6	6	2
$C_3^2.D_4$	3_i	13_+	7_k	0	-6	4	1^4	10_k	3	-2		1^3	3_k	3			42	42	22	6	2	32	6
44.62	3	2.75	1.72	9	-12	12	1^2	3_i	-2			1^3	4_+	1		c	42	42		6	32	32	222
$\widetilde{S_6}$	1_k	11_-	9_k	0	-9	8	1^4	11_-	2	-2	y	1^6	9_k	3	3		5	5	5	6	6	32	6
53.06		4320	2t*10*	0	0	4	2	0_+	-3								5	5		6	32	32	32
$\widehat{S_6}$	3_i	13_-	7_i	0	6	4	1^4	11_i	-2	2	x	1^3	4_+	-1			5	5	5	32	32	6	32
53.06		3.00	1.72	15	12	12	1^2	2_i	3			1^3	3_i	-3			5	5		32	6	6	6

Discriminant class: -3

G	∞	2	**3**	a_1	a_2	a_3	Over $p=2$					Over $p=3$					5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_2	1_k	0_+	1_i	0	3		2	0_+	-3			1^2	1_i	-3			−	+	−	+	−	+	−
1.73		0	*1*														−	+		+	−	+	−
D_4	1_k	11_k	2_-	0	4	0	1^4	11_k	-2	6		1^2	1_i	-3			2	4	22	4	22	22	2
11.65		430	*1*	-2								1^2	1_i	-3			2	4		4	22	22	2
D_4	2_-	11_i	1_i	0	-4	0	1^4	11_i	6	-2		1^2	1_i	-3			22	4	2	4	2	22	22
11.65		2.75	0.50	6													22	4		4	2	22	22
$C_3^2.D_4$	1_k	14_-	7_k	0	-6	8	1^4	11_k	-2	6		1^6	7_k	-3	-3		32	42	6	42	6	22	32
24.24		430	t*10*	-6	0	4	1^2	3_k	6								32	42		42	6	22	32
$C_3^2.D_4$	3_i	14_-	3_i	2	-1	0	1^4	11_i	6	-2		1^3	3_i	-3			6	42	32	42	32	22	6
24.24		2.75	1.17	2	-4	2	1^2	3_i	-2			3	0_+	1			6	42		42	32	22	6
$C_3^2.D_4$	1_k	14_-	11_k	0	-18	12	1^4	11_k	-2	6		1^6	11_k	-3	-3		2	42	222	42	6	22	32
72.71		430	f2*1*	81	-108	-180	1^2	3_k	6								32	42		42	6	22	32
$C_3^2.D_4$	3_i	14_-	9_i	0	6	-4	1^4	11_i	6	-2		1^3	5_i	-3		a	222	42	2	42	32	22	6
72.71	3	2.75	2.17	27	-36	12	1^2	3_i	-2			1^3	4_+	1		a	6	42		42	32	22	6
$C_3^2.D_4$	1_k	14_-	11_k	0	-18	12	1^4	11_k	-2	6		1^6	11_k	-3	-3		32	42	6	42	222	22	32
72.71	3	430	f2*1*	81	-108	12	1^2	3_k	6								2	42		42	222	22	2
$C_3^2.D_4$	3_i	14_-	9_i	0	6	-4	1^4	11_i	6	-2		1^3	5_i	-3		b	6	42	32	42	2	22	6
72.71	3	2.75	2.17	81	36	12	1^2	3_i	-2			1^3	4_+	1		b	222	42		42	2	22	222
$C_3^2.D_4$	1_k	14_-	11_k	0	-18	12	1^4	11_k	-2	6		1^6	11_k	-3	-3		32	42	6	42	6	22	2
72.71		430	f2*1*	27	-36	12	1^2	3_k	6								32	42		42	6	22	32
$C_3^2.D_4$	3_i	14_-	9_i	0	-12	16	1^4	11_i	6	-2		1^3	4_+	1		c	6	42	32	42	32	22	222
72.71	3	2.75	2.17	54	-144	96	1^2	3_i	-2			1^3	5_i	-3		c	6	42		42	32	22	6
$\widetilde{D_4}$	1_k	11_i	2_+	0	-4	0	1^4	11_i	-2	6		2^2	2_+	1		V	2	4	22	4	22	e	2
11.65		430	*10*	-2													2	4		4	22	e	2
D_4	2_-	11_k	1_k	0	4	0	1^4	11_k	6	-2		1^2	1_k	3			22	4	2	4	2	e	22
11.65		2.75	0.50	6								2	0_+	-1			22	4		4	2	e	22
$\widetilde{C_3^2.D_4}$	1_k	14_+	9_i	0	-12	16	1^4	11_i	-2	6		1^6	9_i	-3	-3		32	42	6	42	6	3	32
34.96		430	2*10*	-18	48	-32	1^2	3_k	6								2	42		42	6	33	32
$\widehat{C_3^2.D_4}$	3_i	14_+	5_k	2	1	-4	1^4	11_k	6	-2		1^3	4_+	-1			6	42	32	42	32	33	6
34.96		2.75	1.50	-11	14	33	1^2	3_i	-2			1^2	1_k	3			222	42		42	32	3	6
S_6	1_k	12_-	11_k	0	-9	-24	1^4	10_i	3	-2		1^6	11_k	-3	-3		6	5	32	5	6	22	32
72.71		s320	f2*1*	9	72	3	1^2	2_i	-1								4	5		3	6	5	32
S_6	3_i	12_-	9_i	0	6	-12	1^4	9_k	-2	3		1^3	5_i	-3		c	32	5	6	5	32	22	6
72.71		2.75	2.17	21	0	2	1^2	3_k	6			1^3	4_+	1		c	4	5		33	32	5	6

Discriminant class: -2

G	∞	**2**	3	a_1	a_2	a_3	Over $p=2$ f^e	c_w	d	s	L	Over $p=3$ f^e	c_w	d	s	L	5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6											29	31		37	41	43	47
C_2	1_k	3_i	0_+	0	2		1^2	3_i	-2								−	−	+	−	+	+	−
2.83		3															−	−		−	+	+	−
$\widetilde{D_4}$	1_k	10_i	0_+	0	2	0	1^4	10_i	-1	2		4	0_+	-1	-1		22	2	4	22	22	4	2
6.73		s32	*00*	-1													22	2		22	e	4	2
D_4	2_-	9_-	0_+	0	-2	0	1^4	9_-	2	-1		4	0_+	-1	-1		2	22	4	2	22	4	22
6.73		2.75		2													2	22		2	e	4	22
$\widetilde{C_3^2.D_4}$	1_k	13_i	8_+	0	0	8	1^4	10_i	-1	2		2^3	8_+	1	-1x		222	32	42	6	22	42	32
47.43		s32	22*00*	-18	-24	8	1^2	3_+	2								6	32		6	3	42	32
$\widehat{C_3^2.D_4}$	3_i	11_k	8_+	0	0	8	1^4	9_-	2	-1		2^3	8_+	1	-1y		2	6	42	32	22	42	6
47.43		2.75	1.78	9	12	20	1^2	2_i	-1								32	6		32	33	42	6
D_4	1_k	10_k	2_-	0	-6	0	1^4	10_k	-1	2		2^2	2_-	-1	-1		22	2	4	22	e	4	2
11.65		s32	*10*	-9													22	2		22	22	4	2
D_4	2_-	9_+	2_-	0	6	0	1^4	9_+	2	-1		2^2	2_-	-1	-1		2	22	4	2	e	4	22
11.65	2	2.75	0.50	18													2	22		2	22	4	22
$C_3^2.D_4$	1_k	13_k	6_-	0	-6	4	1^4	10_k	-1	2		2^3	6_-	1	-1x		6	32	42	6	33	42	32
30.94		s32	tt*10*	9	-12	-4	1^2	3_+	2								6	2		222	22	42	32
$C_3^2.D_4$	3_i	11_i	6_-	0	0	0	1^4	9_+	2	-1		2^3	6_-	1	-1y		32	6	42	32	3	42	6
30.94		2.75	1.39	9	-12	4	1^2	2_i	-1								32	222		2	22	42	6

Discriminant class: -1

G	∞	**2**	3	a_1	a_2	a_3	Over $p=2$				Over $p=3$				5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d s	L	f^e	c_w	d s	L	29	31		37	41	43	47
C_2	1_k	2_i	0_+	0	1		1^2	2_i	-1		2	0_+	-1		+	−	−	+	+	−	−
2.00		2	*0*												+	−		+	+	−	−
$\widetilde{D_4}$	1_k	6_+	3_i	2	0	-4	1^4	6_+	-3	3	1^4	3_i	-3	3	4	22	2	22	4	22	2
6.45		220	*110*	-2											4	22		e	4	22	2
D_4	2_-	4_k	3_k	0	0	0	2^2	4_k	3	-3	1^4	3_k	3	-3	4	2	22	22	4	2	22
6.45		1.50	0.75	3											4	2		e	4	2	22
$\widetilde{C_3^2.D_4}$	1_k	8_i	10_+	0	-6	2	1^4	6_+	-3	3	1^6	10_+	-1	3a	42	6	32	22	42	6	32
27.90		220	[9/4]*110*	9	-6	-2	1^2	2_i	3						42	222		33	42	6	32
$\widehat{C_3^2.D_4}$	3_i	4_k	10_+	0	3	-2	2^2	4_k	3	-3	1^6	10_+	-1	-3a	42	32	6	22	42	32	6
27.90		1.50	2.08	9	-12	4	2	0_+	-3						42	2		3	42	32	6
$\widetilde{D_4}$	1_k	8_+	3_i	0	0	0	1^4	8_+	-3	3	1^4	3_i	-3	3	4	22	2	e	4	22	2
9.12		320	*110*	-3											4	22		22	4	22	2
D_4	2_-	6_k	3_k	0	0	0	2^2	6_k	3	-3	1^4	3_k	3	-3	4	2	22	e	4	2	22
9.12		2.00	0.75	12											4	2		22	4	2	22
$\widetilde{C_3^2.D_4}$	1_k	10_i	10_+	0	-6	8	1^4	8_+	-3	3	1^6	10_+	-1	3b	42	6	32	33	42	222	2
39.45		320	[9/4]*110*	9	-24	4	1^2	2_i	3						42	6		22	42	6	32
$\widehat{C_3^2.D_4}$	3_i	6_k	10_+	0	3	-4	2^2	6_k	3	-3	1^6	10_+	-1	-3b	42	32	6	3	42	2	222
39.45		2.00	2.08	9	30	52	2	0_+	-3						42	32		22	42	32	6
$\widetilde{D_4}$	1_k	11_-	3_k	0	0	0	1^4	11_-	-6	6x	1^4	3_k	3	-3	22	22	22	4	4	2	2
18.24		432	*110*	-24											e	22		4	4	2	2
D_4	2_-	11_i	3_i	0	0	0	1^4	11_i	6	-6x	1^4	3_i	-3	3	22	2	2	4	4	22	22
18.24	2	3.00	0.75	6											e	2		4	4	22	22
$\widetilde{C_3^2.D_4}$	1_k	14_i	10_+	0	-18	12	1^4	11_-	-6	6x	1^6	10_+	-1	-3b	22	6	222	42	42	32	32
78.90		432	[9/4]*110*	27	-108	36	1^2	3_k	6						3	222		42	42	32	2
$\widehat{C_3^2.D_4}$	3_i	14_k	10_+	0	-6	4	1^4	11_i	6	-6x	1^6	10_+	-1	3b	22	32	2	42	42	6	6
78.90	2	3.00	2.08	63	60	28	1^2	3_-	-6						33	2		42	42	6	222
$\widetilde{D_4}$	1_k	11_-	3_k	0	0	0	1^4	11_-	-6	6y	1^4	3_k	3	-3	e	22	22	4	4	2	2
18.24		432	*110*	-6											22	22		4	4	2	2
D_4	2_-	11_i	3_i	0	0	0	1^4	11_i	6	-6y	1^4	3_i	-3	3	e	2	2	4	4	22	22
18.24	2	3.00	0.75	24											22	2		4	4	22	22
$\widetilde{C_3^2.D_4}$	1_k	14_i	10_+	0	-6	20	1^4	11_-	-6	6y	1^6	10_+	-1	-3b	33	6	6	42	42	32	32
78.90		432	[9/4]*110*	9	-60	4	1^2	3_k	6						22	6		42	42	32	2
$\widehat{C_3^2.D_4}$	3_i	14_k	10_+	0	12	-16	1^4	11_i	6	-6y	1^6	10_+	-1	3b	3	32	32	42	42	6	6
78.90	2	3.00	2.08	36	-96	160	1^2	3_-	-6						22	32		42	42	6	222

Discriminant class: -1 (continued)

G	∞	**2**	3	a_1	a_2	a_3	Over $p=2$ f^e	c_w	d s L	Over $p=3$ f^e	c_w	d s L	5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d s L	f^e	c_w	d s L	29	31		37	41	43	47
C_2	1_k	2_i	0_+	0	1		1^2	2_i	-1	2	0_+	-1	+	−	−	+	+	−	−
2.00		2	*0*										+	−		+	+	−	−
$\widetilde{D_4}$	1_k	11_i	0_+	0	0	0	1^4	11_i	-2 2x	2	0_+	-1	4	2	22	4	22	22	2
8.00		432	*0*	-2						2	0_+	-1	4	2		4	22	22	2
D_4	2_-	11_-	0_+	0	0	0	1^4	11_-	2 -2x	2	0_+	-1	4	22	2	4	22	2	22
8.00		3.00		2									4	22		4	22	2	22
$\widetilde{C_3^2.D_4}$	1_k	14_i	8_+	0	0	16	1^4	11_i	-2 2x	2^3	8_+	-1 -1	42	32	6	42	22	222	32
56.40		432	22*0*	-18	-48	32	1^2	3_+	2				42	32		42	22	222	2
$\widehat{C_3^2.D_4}$	3_i	14_k	8_+	0	-6	12	1^4	11_-	2 -2x	1^3	4_+	-1	42	6	32	42	22	2	6
56.40	3	3.00	1.78	9	-36	68	1^2	3_i	-2	1^3	4_+	1 a	42	6		42	22	2	222
$\widetilde{D_4}$	1_k	11_i	2_+	0	0	0	1^4	11_i	-2 2y	2^2	2_+	1 V	4	2	22	4	e	22	2
13.86		432	*10*	-18									4	2		4	e	22	2
D_4	2_-	11_-	2_+	0	0	0	1^4	11_-	2 -2y	1^2	1_k	3	4	22	2	4	e	2	22
13.86	2	3.00	0.50	18						1^2	1_i	-3	4	22		4	e	2	22
$\widetilde{C_3^2.D_4}$	1_k	14_i	6_+	0	-6	4	1^4	11_i	-2 2y	2^3	6_+	-1 -1	42	32	222	42	3	6	32
36.79		432	tt*10*	-9	12	-4	1^2	3_+	2				42	32		42	33	6	32
$\widehat{C_3^2.D_4}$	3_i	14_k	6_+	0	6	-4	1^4	11_-	2 -2y	1^3	3_i	-3	42	6	2	42	33	32	6
36.79		3.00	1.39	9	-12	12	1^2	3_i	-2	1^3	3_k	3	42	6		42	3	32	6
$\widetilde{C_3^2.D_4}$	1_k	14_i	10_+	0	-18	36	1^4	11_i	-2 2y	2^3	10_+	-1 -1	42	32	6	42	33	6	32
59.95		432	f*10*	-81	108	36	1^2	3_+	2				42	32		42	3	6	2
$\widehat{C_3^2.D_4}$	3_i	14_k	6_+	2	-1	0	1^4	11_-	2 -2y	1^3	5_i	-3 a	42	6	32	42	3	32	6
59.95		3.00	1.83	10	4	2	1^2	3_i	-2	1^2	1_k	3	42	6		42	33	32	222
$\widetilde{C_3^2.D_4}$	1_k	14_i	10_+	0	-18	36	1^4	11_i	-2 2y	2^3	10_+	-1 -1	42	2	6	42	33	6	32
76.53		432	ft*10*	81	-324	252	1^2	3_+	2				42	32		42	3	6	32
$\widehat{C_3^2.D_4}$	3_i	14_k	8_+	0	6	12	1^4	11_-	2 -2y	1^3	5_i	-3 b	42	222	32	42	3	32	6
76.53		3.00	2.06	27	36	36	1^2	3_i	-2	1^3	3_k	3	42	6		42	33	32	6
$\widetilde{C_3^2.D_4}$	1_k	14_i	10_+	0	-18	12	1^4	11_i	-2 2y	2^3	10_+	-1 -1	42	32	6	42	3	222	2
76.53		432	ft*10*	81	-108	-36	1^2	3_+	2				42	2		42	33	222	32
$\widehat{C_3^2.D_4}$	3_i	14_k	8_+	0	-6	4	1^4	11_-	2 -2y	1^3	3_k	3	42	6	32	42	33	2	222
76.53		3.00	2.06	27	-36	12	1^2	3_i	-2	1^3	5_i	-3 c	42	222		42	3	2	6
$\widetilde{S_6}$	1_k	10_i	10_+	0	-6	8	1^6	10_i	-1	1^6	10_+	-1 -3b	5	4	6	5	33	6	32
49.71		[8/3]2*10*	[9/4]*110*	0	-24	28							42	32		5	5	32	6
$\widehat{S_6}$	3_i	10_k	10_+	0	-6	4	1^4	8_-	-3	1^6	10_+	-1 3b	5	4	32	5	3	32	6
49.71	2	2.33	2.08	27	-48	28	1^2	2_i	3				42	6		5	5	6	32
$\widetilde{S_6}$	1_k	14_i	10_+	0	-6	4	1^4	11_+	2 2y	1^6	10_+	-1 -3a	5	6	32	3	5	6	32
78.90		432	[9/4]*110*	-9	-12	52	1^2	3_i	-2				5	6		42	5	32	6
$\widehat{S_6}$	3_i	14_k	10_+	0	12	4	1^4	11_-	2 2y	1^6	10_+	-1 3a	5	32	6	33	5	32	6
78.90	2	3.00	2.08	27	-12	10	1^2	3_i	-2				5	32		42	5	6	32

Discriminant class: 1

G	∞	2	3	a_1	a_2	a_3	Over $p=2$	Over $p=3$	5	7	11	13	17	19	23
grd	h	SC_2	SC_3	a_4	a_5	a_6	f^e c_w d s L	f^e c_w d s L	29	31		37	41	43	47
$\overline{C_4}$	0_+	11_+	0_+	0	-4	0	1^4 11_+ 2 2x	4 0_+ -1 -1	4	22	4	4	e	4	22
6.73		43	*00*	2					4	e		4	22	4	e
C_4	2_-	11_-	0_+	0	4	0	1^4 11_- 2 2x	4 0_+ -1 -1	4	e	4	4	e	4	e
6.73		43	*00*	2					4	22		4	22	4	22
$C_3^2.C_4$	2_-	14_-	8_+	0	6	-12	1^4 11_- 2 2x	2^3 8_+ 1 -1x	42	3	42	42	33	42	e
47.43		43	22*00*	9	-36	28	1^2 3_+ 2		42	22		42	22	42	22
$C_3^2.C_4$	2_-	14_-	8_+	0	0	-8	1^4 11_- 2 2x	2^3 8_+ 1 -1y	42	33	42	42	3	42	e
47.43		2.75	1.78	-18	-48	-16	1^2 3_+ 2		42	22		42	22	42	22
C_4	2_-	11_+	2_-	0	12	0	1^4 11_+ 2 2y	2^2 2_- -1 -1	4	22	4	4	22	4	e
11.65	2	43	*10*	18					4	e		4	e	4	22
$C_3^2.C_4$	2_-	14_+	6_-	0	6	4	1^4 11_+ 2 2y	2^3 6_- 1 -1x	42	22	42	42	22	42	3
30.94		43	tt*10*	-9	-12	-4	1^2 3_+ 2		42	33		42	33	42	22
$C_3^2.C_4$	2_-	14_+	6_-	0	6	-4	1^4 11_+ 2 2y	2^3 6_- 1 -1y	42	22	42	42	22	42	33
30.94		2.75	1.39	9	-12	-4	1^2 3_+ 2		42	3		42	3	42	22
C_4	0_+	11_-	2_-	0	-12	0	1^4 11_- 2 2y	2^2 2_- -1 -1	4	e	4	4	22	4	22
11.65		43	*10*	18					4	22		4	e	4	e
A_6	2_-	10_-	8_+	0	-3	12	1^6 10_- 1	2^3 8_+ 1 -1x	5	5	3	42	5	42	5
31.66		[8/3]*10*	22*00*	-9	0	1			5	42		5	5	5	42
A_6	2_-	8_-	8_+	3	3	2	1^4 8_- -3	2^3 8_+ 1 -1y	5	5	33	42	5	42	5
31.66		2.17	1.78	-3	-3	-1	2 0_+ -3		5	42		5	5	5	42
A_6	2_-	12_+	10_-	0	-3	12	1^4 10_k -1 -6	1^3 5_i -3 a	5	42	5	5	5	42	5
64.35		s32	ft*1*	-6	0	2	1^2 2_i -1	1^3 5_i -3 b	3	42		42	42	5	42
A_6	2_-	12_+	8_-	0	6	-4	1^4 9_- -6 -1	1^3 3_i -3	5	42	5	5	5	42	5
64.35		2.75	2.06	-3	-12	-12	1^2 3_- -6	1^3 5_i -3 c	33	42		42	42	5	42
A_6	2_-	12_+	10_-	0	6	-12	1^4 10_k 3 -2	1^3 5_i -3 b	5	5	42	42	5	42	5
64.35		s32	ft*1*	-15	36	-40	1^2 2_i 3	1^3 5_i -3 c	5	42		5	3	5	42
A_6	2_-	12_+	8_-	0	0	8	1^4 9_k -2 3	1^3 3_i -3	5	5	42	42	5	42	5
64.35		2.75	2.06	9	0	-6	1^2 3_i -2	1^3 5_i -3 a	5	42		5	33	5	42
A_6	2_-	14_+	10_-	0	6	12	1^4 11_- -6 6y	1^3 5_i -3 b	42	5	5	42	5	5	3
76.53		432	ft*1*	57	36	-4	1^2 3_- -6	1^3 5_i -3 a	5	5		3	5	42	22
A_6	2_-	14_+	8_-	0	0	-12	1^4 11_i 6 -6y	1^3 3_i -3	42	5	5	42	5	5	33
76.53		3.00	2.06	21	12	-34	1^2 3_k 6	1^3 5_i -3 c	5	5		33	5	42	22

There are many patterns to be found on Table 6. Here we just make one comment about the solvable sextics and one about the non-solvable sextics.

New solvable sextics. This comment is point-by-point analogous to the comment made in §1.3. For the thirty-two $G = C_4$ or D_4 quartic fields F, the 2-decomposition group $\mathrm{Gal}(F_2)$ is always G itself. Up to conjugation the only subgroups of $C_3^2.G$ surjecting onto G are $C_3^2.G$ itself and $G \subset C_3^2.G$. Since there is no surjection from $\mathrm{Gal}(\overline{\mathbb{Q}}_2/\mathbb{Q}_2)$ onto $C_3^2.G$, the 2-decomposition group $\mathrm{Gal}(K_2)$ for all the new solvable sextics K is just G. In fact $K_2 \cong F_2 \times L_2$ where L_2 is the unique quadratic subfield of F_2. This accounts for the repetitiveness of the 2-local block of columns.

New non-solvable sextics. The tables show that the 62 new non-solvable sextics, besides being unramified outside $\{\infty, 2, 3\}$, are all ramified at ∞, and all wildly ramified at 2 and 3. In fact, at 2 the largest possible order of the wild inertia group P_2 allowed by local conditions is 8. For 30 of the 31 pairs one has $|P_2| = 8$; the lone exception is our first-listed A_6 pair which has $|P_2| = 4$. At 3, the largest possible order of P_3 is 9. Again for 30 of the 31 twin pairs in fact $|P_3| = 9$; the lone exception is our first-listed $d = 6$ pair with $|P_3| = 3$.

3.4. Completeness by class field theory. Let F be one of the twenty-eight D_4 quartics or one of the four C_4 quartics with discriminant $-^j 2^a 3^b$. Here we use class field theory to determine the number of $C_3^2.D_4$ or $C_3^2.C_4$ sextics with discriminant $-^j 2^a 3^b$ and resolvent quartic F. The results prove that our computer search, which was not guaranteed to find all the $C_3^2.C_4$ and $C_3^2.D_4$ sextics, in fact did.

Let $L \subset F$ be the unique quadratic subfield and $\sigma \in \mathrm{Gal}(F/L)$ be the involution fixing exactly L. Let $\widetilde{K}$ be a maximal abelian pro-3-extension of F ramified only in $S = \{\infty, 2, 3\}$. The field $\widetilde{K}$ is the compositum over F of two disjoint subfields $\widetilde{K}^+$ and $\widetilde{K}^-$. Here σ acts trivially on $G_3^+ := \mathrm{Gal}(\widetilde{K}^+/F)$ and by inversion on $G_3^- := \mathrm{Gal}(\widetilde{K}^-/F)$.

Let $\tilde{\sigma} \in \mathrm{Gal}(\widetilde{K}^-/L)$ be any lift of $\sigma \in \mathrm{Gal}(F/L)$. Let $H \subset G_3^-$ be a subgroup of index 3. For clarity we remark that $[\widetilde{K}^H : \mathbb{Q}] = 12$. However more to the point, one has $[\widetilde{K}^{H,\tilde{\sigma}} : \mathbb{Q}] = 6$. The map

$$\{\text{index 3 subgroups of } G_3^-\} \to \{\text{Sextic fields with resolvent quartic } F\}$$
$$H \mapsto \widetilde{K}^{H,\tilde{\sigma}}$$

is always surjective. It is moreover 1-to-1 if $\mathrm{Gal}(F) = D_4$ and 2-to-1 if $\mathrm{Gal}(F) = C_4$.

We are thus reduced to describing the abelian group G_3^- modulo cubes. In fact, in each case we will describe G_3^- itself. Let $U_3^{(1)} \subset F_3^\times$ be the group of 3-adic 1-units. Let $E \subset F^\times$ be the group of global units, $E^{(1)}$ the subgroup of global units which are 1-units at 3, and finally $E_3^{(1)}$ its 3-adic completion. The involution $\sigma \in \mathrm{Aut}(F/\mathbb{Q})$ acts, decomposing both $E_3^{(1)}$ and $U_3^{(1)}$ into a $(+)$-eigenspace and a $(-)$-eigenspace.

For each of our quartic C_4 or D_4 base fields, the class number is prime to 3 as can be seen from the tables in §3.3. There are two D_4 fields where the 2-adic units have non-trivial pro-3 completion, namely those with $f = 2$ and $e = 2$ at 2. Here the pro-3 completion is just $\mathbb{F}_4^\times \cong \mathbb{Z}/3$. However the quadratic subfield L has $f = 2$ and $e = 1$, and so this $\mathbb{Z}/3$ contributes to G_3^+ and not G_3^-.

Discriminant class: 2

G	∞	2	3	a_1	a_2	a_3	Over $p=2$			Over $p=3$			5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d s L	f^e	c_w	d s L	29	31		37	41	43	47
C_2	0_+	3_+	0_+	0	-2		1^2	3_+	2	2	0_+	-1	−	+	−	−	+	−	+
2.83		3	*1*										−	+		−	+	−	+
$\widetilde{D_4}$	0_+	10_i	3_k	0	-6	0	1^4	10_i	3 6	1^4	3_k	3 -3	2	4	22	22	4	2	22
15.33		s32	*110*	3									2	4		22	4	2	22
$\widetilde{D_4}$	0_+	9_k	3_i	0	-6	0	1^4	9_k	6 3	1^4	3_i	-3 3	22	4	2	2	4	22	22
15.33		2.75	0.75	6									22	4		2	4	22	22
D_4	2_-	10_k	3_k	0	6	0	1^4	10_k	3 6	1^4	3_k	3 -3	2	4	22	22	4	2	e
15.33	2	s32	*110*	3									2	4		22	4	2	e
D_4	2_-	9_i	3_i	0	6	0	1^4	9_i	6 3	1^4	3_i	-3 3	22	4	2	2	4	22	e
15.33	2	2.75	0.75	6									22	4		2	4	22	e
$C_3^2.D_4$	2_-	13_-	6_+	0	0	-4	1^4	10_k	3 6	1^6	6_+	-1 -3	32	42	6	6	42	32	3
24.99		s32	[5/4]*110*	-3	-12	-2	1^2	3_k	6				32	42		6	42	32	33
$C_3^2.D_4$	2_-	11_-	6_+	0	3	-4	1^4	9_i	6 3	1^6	6_+	-1 3	6	42	32	32	42	6	33
24.99		2.75	1.19	0	0	-2	1^2	2_i	3				6	42		32	42	6	3
$C_3^2.D_4$	2_-	13_-	10_+	0	-6	16	1^4	10_k	3 6	1^6	10_+	-1 -3a	2	42	6	6	42	32	33
66.35		s32	[9/4]*110*	-45	24	40	1^2	3_k	6				2	42		222	42	32	3
$C_3^2.D_4$	2_-	11_-	10_+	0	-6	8	1^4	9_i	6 3	1^6	10_+	-1 3a	222	42	32	32	42	6	3
66.35		2.75	2.08	-18	48	-32	1^2	2_i	3				222	42		2	42	6	33
$C_3^2.D_4$	2_-	13_-	10_+	0	12	16	1^4	10_k	3 6	1^6	10_+	-1 -3b	32	42	222	6	42	2	3
66.35	2	s32	[9/4]*110*	36	96	-32	1^2	3_k	6				32	42		6	42	2	33
$C_3^2.D_4$	2_-	11_-	10_+	0	-6	4	1^4	9_i	6 3	1^6	10_+	-1 3b	6	42	2	32	42	222	33
66.35		2.75	2.08	9	-12	-44	1^2	2_i	3				6	42		32	42	222	3
$C_3^2.D_4$	2_-	13_-	10_+	0	-6	4	1^4	10_k	3 6	1^6	10_+	-1 -3c	32	42	6	222	42	32	33
66.35		s32	[9/4]*110*	9	-12	-20	1^2	3_k	6				32	42		6	42	32	3
$C_3^2.D_4$	2_-	11_-	10_+	0	12	-8	1^4	9_i	6 3	1^6	10_+	-1 3c	6	42	32	2	42	6	3
66.35		2.75	2.08	9	-12	4	1^2	2_i	3				6	42		32	42	6	33
S_6	2_-	7_-	10_+	0	3	-4	1^4	4_+	-3	1^6	10_+	-1 -3c	6	5	4	6	5	32	5
41.80		3[4/3]*10*	[9/4]*110*	0	-6	-2	1^2	3_-	-6				32	5		6	33	6	42
S_6	2_-	11_-	10_+	0	-6	4	1^6	11_-	2	1^6	10_+	-1 3c	32	5	4	32	5	6	5
41.80		2.08	2.08	18	-12	-26							6	5		32	3	32	42
S_6	2_-	11_-	8_+	0	-6	4	1^6	11_-	2	1^3	4_+	1 c	6	5	6	32	42	4	3
42.25		3[8/3]*10*	22*0*	6	0	-6				1^3	4_+	-1	32	5		6	5	4	42
S_6	2_-	11_-	8_+	0	-6	4	1^4	8_+	-3	2^3	8_+	-1 -1	32	5	32	6	42	4	33
42.25		2.58	1.78	6	0	-4	1^2	3_-	-6				6	5		32	5	4	42

Discriminant class: 2 (continued)

G	∞	2	3	a_1	a_2	a_3	Over $p=2$					Over $p=3$					5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_2	0_+	3_+	0_+	0	-2		1^2	3_+	2			2	0_+	-1			−	+	−	−	+	−	+
2.83		3	*1*														−	+		−	+	−	+
S_6	2_-	9_-	10_+	0	3	6	1^4	6_-	1			2^3	10_+	-1	-1		6	5	32	4	5	32	42
45.51		322*0*	ft*10*	12	-36	-2	1^2	3_+	2								32	42		32	5	32	5
S_6	2_-	9_-	8_+	0	0	2	3^2	9_-	2			1^3	5_i	-3		c	32	5	6	4	5	6	42
45.51		2.25	2.06	-3	-6	-3						1^3	3_k	3			6	42		6	5	6	5
S_6	2_-	9_-	10_+	0	-9	6	1^4	6_-	1			1^6	10_+	-1	3	b	4	42	6	4	5	32	42
46.92		322*0*	[9/4]*110*	0	0	-18	1^2	3_+	2								6	5		6	5	32	42
S_6	2_-	9_-	10_+	0	-6	10	3^2	9_-	2			1^6	10_+	-1	-3	b	4	42	32	4	5	6	42
46.92		2.25	2.08	-9	6	1											32	5		32	5	6	42
S_6	2_-	11_-	8_+	0	0	8	1^6	11_-	2			1^3	5_i	-3		b	6	42	4	32	5	6	42
57.33		3[8/3]*10*	ft*10*	-12	12	-6						1^3	3_k	3			4	5		4	5	32	33
S_6	2_-	11_-	10_+	0	-6	12	1^4	8_+	-3			2^3	10_+	-1	-1		32	42	4	6	5	32	42
57.33		2.58	2.06	-6	0	4	1^2	3_-	-6								4	5		4	5	6	3
S_6	2_-	13_-	10_+	0	-6	16	1^4	10_i	-1	-6		1^6	10_+	-1	-3	c	6	5	6	6	3	4	5
66.35		s320	[9/4]*110*	-18	24	4	1^2	3_i	-2								4	3		32	42	4	5
S_6	2_-	9_-	10_+	0	-6	8	1^4	9_-	-6	-1		1^6	10_+	-1	3	c	32	5	32	32	33	4	5
66.35		2.75	2.08	18	-24	-32	2	0_+	-3								4	33		6	42	4	5
S_6	2_-	11_-	10_+	0	-9	-12	1^4	9_i	-2	3		1^6	10_+	-1	-3	b	6	5	6	6	5	4	42
66.35	2	s320	[9/4]*110*	0	0	-18	1^2	2_i	-1								32	5		4	22	32	3
S_6	2_-	13_-	10_+	0	12	-16	1^4	10_k	3	-2		1^6	10_+	-1	3	b	32	5	32	32	5	4	42
66.35	2	2.75	2.08	-9	-96	-2	1^2	3_k	6								6	5		4	22	6	33
S_6	2_-	11_-	8_+	3	9	14	1^4	11_-	-6	6	x	1^3	5_i	-3		a	6	5	6	6	33	4	3
76.53		4320	ft*10*	3	3	3	2	0_+	-3			1^3	3_k	3			6	22		6	42	6	42
S_6	2_-	13_-	10_+	0	-6	12	1^4	11_i	6	-6	y	2^3	10_+	-1	-1		32	5	32	32	3	4	33
76.53		3.00	2.06	-15	36	-8	1^2	2_i	3								32	22		32	42	32	42
S_6	2_-	13_-	8_+	0	-3	8	1^4	11_i	-2	6		1^3	5_i	-3		a	6	5	32	6	5	32	5
76.53		4320	ft*10*	0	0	12	1^2	2_i	-1			1^3	3_k	3			6	42		32	5	4	3
S_6	2_-	13_-	10_+	0	-6	12	1^4	11_i	6	-2		2^3	10_+	-1	-1		32	5	6	32	5	6	5
76.53		3.00	2.06	3	-36	28	1^2	2_i	3								32	42		6	5	4	33
S_6	2_-	13_-	8_+	0	-6	4	1^4	11_i	-2	2	y	1^3	3_k	3			4	42	6	32	42	4	42
76.53		432	ft*10*	9	-36	-24	1^2	2_i	-1			1^3	5_i	-3		b	6	5		6	5	6	42
S_6	2_-	11_-	10_+	-3	6	8	1^4	11_-	2	-2	y	2^3	10_+	-1	-1		4	42	32	6	42	4	42
76.53		3.00	2.06	-30	42	-20											32	5		32	5	32	42

Discriminant class: 2 (continued)

G	∞	2	3	a_1	a_2	a_3	Over $p=2$ f^e	c_w	d s L	Over $p=3$ f^e	c_w	d s L
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d s L	f^e	c_w	d s L
C_2	0_+	3_+	0_+	0	-2		1^2	3_+	2	2	0_+	-1
2.83		3	*1*									
S_6	2_-	13_-	8_+	0	3	8	1^4	11_i	6 -2	1^3	5_i	-3 c
76.53		4320	ft*10*	-27	48	-29	1^2	2_i	3	1^3	3_k	3
S_6	2_-	13_-	10_+	0	-9	-24	1^4	11_i	-2 6	2^3	10_+	-1 -1
76.53		3.00	2.06	-27	0	-9	1^2	2_i	-1			
S_6	2_-	11_-	10_+	-3	0	4	1^4	11_-	2 -2y	1^6	10_+	-1 -3c
78.90		432	[9/4]*110*	12	-36	16						
S_6	2_-	13_-	10_+	0	-6	20	1^4	11_i	-2 2y	1^6	10_+	-1 3c
78.90		3.00	2.08	-9	12	-20	1^2	2_i	-1			
S_6	2_-	11_-	10_+	-3	9	-14	1^4	11_-	-6 6y	1^6	10_+	-1 3c
78.90		4320	[9/4]*110*	21	-9	1	2	0_+	-3			
S_6	2_-	13_-	10_+	0	3	8	1^4	11_i	6 -6x	1^6	10_+	-1 -3c
78.90		3.00	2.08	27	-24	-71	1^2	2_i	3			

G / grd	5 / 29	7 / 31	11	13 / 37	17 / 41	19 / 43	23 / 47
C_2	−	+	−	−	+	−	+
2.83	−	+		−	+	−	+
S_6	4	42	6	6	5	6	33
76.53	32	5		4	5	32	5
S_6	4	42	32	32	5	32	3
76.53	6	5		4	5	6	5
S_6	6	5	4	32	5	6	42
78.90	32	5		6	42	4	5
S_6	32	5	4	6	5	32	42
78.90	6	5		32	42	4	5
S_6	6	33	6	32	5	4	5
78.90	32	3		32	42	32	5
S_6	32	3	32	6	5	4	5
78.90	6	33		6	42	6	5

Discriminant class: 3

G	∞	**2**	**3**	a_1	a_2	a_3	Over $p=2$ f^e	c_w	d s L	Over $p=3$ f^e	c_w	d s L
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d s L	f^e	c_w	d s L
C_2	0_+	2_i	1_k	0	-3		1^2	2_i	3	1^2	1_k	3
3.46		2	*1*									
S_6	2_-	10_i	9_i	0	-3	4	1^6	10_i	3	1^3	4_+	-1
54.47		[8/3]2*10*	f2*10*	9	0	-9				1^3	5_i	-3 a
S_6	2_-	10_k	11_k	0	-9	24	1^4	8_-	-3	1^6	11_k	3 3
54.47		2.33	2.17	27	-72	69	1^2	2_i	-1			
S_6	2_-	12_i	9_i	0	6	-12	1^4	9_+	-6 -1	1^3	5_i	-3 c
72.71		s320	f2*10*	-15	0	2	1^2	3_i	-2	1^3	4_+	-1
S_6	2_-	10_k	11_k	-3	-3	14	1^4	10_k	-1 -6	1^6	11_k	3 3
72.71		2.75	2.17	-39	51	-23	2	0_+	-3			
S_6	2_-	14_k	11_k	0	0	-36	1^4	11_k	6 -2	1^6	11_k	3 3
86.47		4320	f2*10*	63	-36	6	1^2	3_+	2			
S_6	2_-	14_i	9_i	0	-6	-16	1^4	11_k	-2 6	1^3	4_+	-1
86.47		3.00	2.17	6	0	-12	1^2	3_-	-6	1^3	5_i	-3 a
S_6	2_-	14_i	9_i	0	6	24	1^4	11_-	2 -2y	1^3	4_+	-1
86.47		4320	f2*10*	30	0	-4	1^2	3_k	6	1^3	5_i	-3 c
S_6	2_-	14_k	11_k	0	-18	48	1^4	11_i	-2 2x	1^6	11_k	3 3
86.47	2	3.00	2.17	-18	0	-12	1^2	3_-	-6			

G / grd	5 / 29	7 / 31	11	13 / 37	17 / 41	19 / 43	23 / 47
C_2	−	−	+	+	−	−	+
3.46	−	−		+	−	−	+
S_6	4	6	5	42	6	4	5
54.47	6	6		22	32	32	5
S_6	4	32	5	42	32	4	5
54.47	32	32		22	6	6	5
S_6	6	6	5	5	32	32	33
72.71	32	6		3	6	6	22
S_6	32	32	5	5	6	6	3
72.71	6	32		33	32	32	22
S_6	6	6	3	42	6	4	42
86.47	2	4		5	6	6	5
S_6	32	32	33	42	32	4	42
86.47	222	4		5	32	32	5
S_6	6	32	42	3	32	4	5
86.47	32	4		42	6	6	5
S_6	32	6	42	33	6	4	5
86.47	6	4		42	32	32	5

Discriminant class: 6

G	∞	**2**	**3**	a_1	a_2	a_3	Over $p=2$					Over $p=3$					5	7	11	13	17	19	23
grd	h	sc_2	sc_3	a_4	a_5	a_6	f^e	c_w	d	s	L	f^e	c_w	d	s	L	29	31		37	41	43	47
C_2	0_+	3_k	1_i	0	-6		1^2	3_k	6			1^2	1_i	-3			+	−	−	−	−	+	+
4.90		3	*1*														+	−		−	−	+	+
S_6	2_-	11_i	5_i	-2	1	0	1^6	11_i	6			1^3	5_i	-3		a	5	6	6	6	6	5	42
44.91		3[8/3]*1*0	f*1*0	1	-2	-1						3	0_+	1			33	4		32	6	42	42
S_6	2_-	11_k	11_k	0	0	12	1^4	8_-	-3			1^6	11_k	-3	-3		5	32	32	32	32	5	42
44.91		2.58	1.83	27	0	-18	1^2	3_i	-2								3	4		6	32	42	42
S_6	2_-	11_i	9_i	0	-6	4	1^4	8_+	-3			1^3	4_+	1		c	5	6	32	6	6	5	5
64.78		3[8/3]*1*0	f2*1*	18	24	4	1^2	3_i	-2			1^3	5_i	-3		c	5	2		6	6	42	5
S_6	2_-	11_k	11_k	0	0	12	1^6	11_k	6			1^6	11_k	-3	-3		5	32	6	32	32	5	5
64.78		2.58	2.17	54	0	-18											5	222		32	32	42	5
S_6	2_-	13_i	9_i	0	6	16	1^4	10_i	3	6		1^3	5_i	-3		c	5	6	32	32	4	22	42
72.71		s32	f2*1*	18	72	-12	1^2	3_+	2			1^3	4_+	1		b	5	32		32	32	42	5
S_6	2_-	9_k	11_k	-3	6	-10	1^4	9_k	6	3		1^6	11_k	-3	-3		5	32	6	6	4	22	42
72.71		2.75	2.17	6	6	-8											5	6		6	6	42	5
S_6	2_-	13_i	9_i	0	-12	4	1^4	10_k	-1	2		1^3	5_i	-3		c	5	6	6	4	6	5	3
72.71	2	s320	f2*1*	45	-36	-6	1^2	3_-	-6			1^3	4_+	1		a	42	6		32	32	5	42
S_6	2_-	11_k	11_k	0	-9	12	1^4	9_-	2	-1		1^6	11_k	-3	-3		5	32	32	4	32	5	33
72.71	2	2.75	2.17	0	0	-6	1^2	2_i	3								42	32		6	6	5	42
S_6	2_-	11_i	9_i	0	-3	8	1^4	11_i	-2	2y		1^3	5_i	-3		b	5	6	4	6	32	42	5
86.47		4320	f2*1*	9	0	9	2	0_+	-3			1^3	4_+	1		a	42	32		32	4	5	42
S_6	2_-	13_k	11_k	0	0	24	1^4	11_-	2	-2x		1^6	11_k	-3	-3		5	32	4	32	6	42	5
86.47		3.00	2.17	54	0	-72	1^2	2_i	3								42	6		6	4	5	42
S_6	2_-	13_i	9_i	0	3	8	1^4	11_+	-6	-6y		1^3	5_i	-3		c	42	6	6	4	4	5	33
86.47		432	f2*1*	-3	0	3	1^2	2_i	-1			1^3	4_+	1		b	42	4		32	6	22	5
S_6	2_-	13_k	11_k	0	-9	24	1^4	11_-	-6	-6y		1^6	11_k	-3	-3		42	32	32	4	4	5	3
86.47		3.00	2.17	27	-72	-123	1^2	2_i	-1								42	4		6	32	22	5

Class field theory gives an idelic description of G_3^-. Because of the vanishing described in the previous paragraph, this idelic description reduces to the 3-adic description

$$G_3^- = \mathrm{Coker}(\phi\colon E_3^{(1)-} \to U_3^{(1)-}).$$

In each case, one knows $E_3^{(1)-}$ abstractly from Dirichlet's theorem. Similarly, it is easy to determine $U_3^{(1)-}$ abstractly from the D_4 and C_4 entries in the tables in §3.3. Also, it follows from general principles that the groups $E_3^{(1)-}$, $U_3^{(1)-}$, and $\mathrm{Coker}(\phi)$ are isomorphic for a dihedral quartic field F and its twin F^t.

For the thirty-two ground fields F in question, only four pairs of abstract groups $(E_3^{(1)-}, U_3^{(1)-})$ arise, according to which we divide into cases A–D:

	$E_3^{(1)-}$	$U_3^{(1)-}$	Minimal Coker	$\lvert C_3^2.D_4$ Fields$\rvert$	$\lvert C_3^2.C_4$ Fields$\rvert$
A	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$	$\{0\}$	$\mathbf{0}, 1, 4$	$\mathbf{0}, 2$
B	$\mathbb{Z}_3$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3$	$\mathbf{1}, 4$	
C	$\mathbb{Z}_3$	$\mathbb{Z}_3^2 + \mathbb{Z}/3$	$\mathbb{Z}_3 + \mathbb{Z}/3$	$\mathbf{4}, 13$	
D	$\{0\}$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$	$\mathbf{4}$	$\mathbf{2}$

The four cases match up with the C_4 and D_4 ground fields as they are listed in §3.3 as shown here:

-6	-3	-2	-1	1	2
C	C	B	B	A	**A**
B	B	B	B	D	D
			B	D	
			B	A	
			B		
			C		

Thus the first twin pair of D_4 quartics for $d = -6$ lie in Case C and so on. The cases worked out below correspond to the entries in boldface.

The entries in the $|C_3^2.D_4$ Fields$|$ column give the number of $C_3^2.D_4$ fields corresponding to various possible cokernels. The smallest number is printed in boldface; this is the number of $C_3^2.D_4$ fields if and only if the map ϕ is as surjective as possible, yielding the minimal cokernel tabulated. Similar remarks apply to the $|C_3^2.C_4$ Fields$|$ although there are fewer possibilities.

Note that the number of fields found by the computer search is always the number in boldface. Thus, proving that the table is complete is equivalent to proving the following proposition.

Proposition 3.1. *For each of the* 32 *quartic ground fields F in question, the cokernel of ϕ: $E_3^{(1)-} \to U_3^{(1)-}$ is minimal.*

Of course, in Case D there is nothing to prove. As remarked, for the dihedral cases A–C we have only to prove the statement either for F or F^t. This leaves 15 cases, which we considered separately.

In each of these cases, we presented F by an even polynomial

$$F = \mathbb{Q}[x]/(x^4 + ax^2 + b).$$

The involution σ then acts simply by $\sigma(x) = -x$. We then used Pari to find one or two global units in the group $E_3^{(1)-}$, either u_1 in Cases B and C or u_1, u_2 in Case A. We then verified that u_1 or u_1, u_2, u_1u_2, u_1/u_2 are not cubes in $U_3^{(1)-}$. For this last step it suffices to work modulo a sufficiently large power of 3. Note, Pari can be pushed to guarantee a basis for global units, and the u_i we obtained most likely form a basis of $E_3^{(1)-}$. However, in this case it is unnecessary to know the value of ϕ applied to a basis of $E_3^{(1)-}$. We need only know that there exist global units which are not cubes, as described above. We now outline two typical calculations.

Typical Case C computation. Consider the first pair of D_4 twins in the table for discriminant class -6. We worked with $F = \mathbb{Q}[x]/(x^4 - 2x^2 + 3)$. Here $\mathcal{O}_F = \mathbb{Z}[x]$. Both F and its quadratic subfield are totally imaginary so E_3^+ and E_3^- have ranks 0 and 1 respectively, the latter group containing $v = x^3 + x^2 - x - 2$. One has the factorization $x^4 - 2x^2 + 3 = (x^2 + 3)(x^2 + 4) \in \mathcal{O}_F/9$. This corresponds to the factorization into quadratic rings $\mathcal{O}_{F_3} = \mathbb{Z}_3[\sqrt{-3}] \times \mathbb{Z}_3[\sqrt{-4}]$ given on the tables. The factor $\mathbb{Z}_3[\sqrt{-3}]$ has third roots of unity and thus contributes $\mathbb{Z}_3 \times \mathbb{Z}/3$ to $U_3^{(1)-}$. On the other hand the factor $\mathbb{Z}_3[\sqrt{-4}]$ contributes $\mathbb{Z}_3$ to $U_3^{(1)-}$. Thus indeed we are in Case C.

The residue fields for $\mathbb{Z}_3[\sqrt{-3}]$ and $\mathbb{Z}_3[\sqrt{-4}]$ are $\mathbb{F}_3$ and $\mathbb{F}_9$ respectively. Thus, to insure we have a 1-unit, we let $u = v^8$ and so

$$\begin{aligned} u &= -4904x^3 - 9024x^2 - 2792x + 4225 \\ &\equiv x^3 + 3x^2 + 7x + 4 \pmod 9. \end{aligned}$$

Let $\bar{x}$ be the image of x in the unramified factor $\mathbb{Z}_3[\sqrt{-4}]$; so $\bar{x}$ is a unit and satisfies the congruence $\bar{x}^2 + 4 \equiv 0 \pmod 9$. The image of u in $\mathbb{Z}_3[\sqrt{-4}]$ is the 1-unit $3\bar{x} + 1$. Since 3 is a uniformizer, the cube of any 1-unit in $\mathbb{Z}_3[\sqrt{-4}]$ is congruent to 1 modulo 9. Hence $3\bar{x} + 1$ is not a cube in $\mathbb{Z}_3[\sqrt{-4}]$, and we conclude that $u \in E_3^{(1)-}$ is not a cube in $U_3^{(1)-}$.

Typical Case A computation. Consider the first D_4 field from the tables with discriminant class 2. Then $f(x) = x^4 - 6x^2 + 3$, an Eisenstein polynomial for $p = 3$. Thus, $\mathcal{O}_{F_3} = \mathbb{Z}_3[x]$ and x is a local uniformizer. Since F is totally real, E_3^+ has rank 1 and E_3^- has rank 2. The group E_3 contains $v_1 = x^3 + x^2 - 5x - 4$, $v_2 = x^3 + 2x^2 - x - 1$, and $v_3 = x^3 - 2x^2 - x + 1$. It is easy to check that $\sigma(v_1)v_1 = 1$. We then anti-symmetrize v_2 to produce $v_2/\sigma(v_2) = -2x^3 - 4x^2 + 2x + 1$. Taking squares to produce 1-units (since the residual degree is clearly 1), we have $u_1 = v_1^2 = -6x^3 - 4x^2 + 34x + 25$ and $u_2 = (v_2/\sigma(v_2))^2 = 76x^3 + 176x^2 - 44x - 95$.

In this case we work modulo x^6. A brief computation shows that elements of $\mathcal{O}_{F_3}/(x^6)$ can be written uniquely in the form $ax^3 + bx^2 + cx + d$ where a and b are taken modulo 3, and c and d are taken modulo 9. It is easy to compute the cube of this expression; cubes are of the form $c^3x^3 + 3cd^2x + d^3$. Reducing the units above modulo x^6, we find

$$\begin{aligned} u_1 &\equiv 2x^2 + 7x + 7 \\ u_2 &\equiv x^3 + 2x^2 + x + 4 \\ u_1u_2 &\equiv 2x^3 + 2x^2 + 2x + 4 \\ u_1/u_2 &\equiv 2x^3 + 1. \end{aligned}$$

Thus each expression is not a cube mod x^6, hence not a cube in $U_3^{(1)-}$.

Added in proof: We have now incorporated into our programs the improvements described in §2.5. We have obtained analogous complete lists for sextics ramified within other small sets of primes. The case of $\{\infty, 2, 3\}$ septics also has been completed. Tables, similar in format to Table 1 and Table 6, are available at [**J**].

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