Stable and Unstable Manifolds for Planar Dynamical Systems

H.L. Smith*
Department of Mathematics
Arizona State University
Tempe, AZ 85287–1804

October 20, 2008

Key words and phrases: Saddle point, stable and unstable manifolds

AMS subject classifications: 34C30

1 Introduction

An introduction to the qualitative study of planar systems of nonlinear ordinary differential equations is typically encountered at the junior or senior year in a second course in differential equations. These courses are becoming increasingly popular, due to the widespread use of inexpensive computer software which allows the student to experiment very early on in the learning process and because dynamical systems ideas have recently permeated many other disciplines. Traditionally, such courses introduce the contraction mapping theorem to establish the existence of solutions to initial value problems. Linear systems, stability and linearized stability analysis are usually emphasized. The Poincare-Bendixson theorem may be stated but is rarely proved. See the text of Waltman [5] for a typical treatment. More recent texts have emphasized the dynamical systems point of view and even touch on chaos and bifurcation theory at the expense of more traditional topics.

*Supported by NSF Grant DMS 9700910

The author is unaware of any text at this level which treats the existence of stable and unstable manifolds corresponding to saddle point equilibria despite the fact that these results can be easily obtained from the contraction mapping theorem and despite the obvious importance of these special solutions. Indeed, it is hard to over-emphasize the importance of the stable and unstable manifolds of saddle equilibria to understanding the global behavior of the orbits of a planar system. They form separatrices, partitioning the plane into invariant regions of differing dynamics. Often they form basin boundaries, that is, boundaries separating the domains of attraction of different attractors. Unstable manifolds usually lead to attracting equilibria or periodic orbits.

Consider, for example, the Lotka-Volterra system

$$\begin{aligned} x' &= x(r_1 - a_{11}x - a_{12}y) \\ y' &= y(r_2 - a_{21}x - a_{22}y) \end{aligned}$$

in the case of strong interspecific competition:

$$a_{11}a_{22} < a_{21}a_{12}.$$

Figure 1 depicts this well-known phase diagram. The positive equilibrium in the interior of the first quadrant is a saddle point, that is, the Jacobian matrix of the vector field at the point has real eigenvalues of opposite sign. As a consequence of the main result of this paper, there are two orbits, parameterized by solutions approaching the positive equilibrium point at the same exponential rate as $t \to \infty$, which together with the equilibrium point itself, form the stable manifold of that point. In Figure 1, the manifold joins the origin with the point at infinity forming a monotone curve. In addition, there are two orbits, parameterized by solutions approaching the positive equilibrium at the same exponential rate as $t \to -\infty$, which together with the equilibrium, form the unstable manifold of the equilibrium. These orbits approach the single-population equilibria on the boundary. However, the stable manifold is the dominant feature of the phase plane because it separates the positive quadrant into the domains of attraction of the two single-population equilibria. Competitive exclusion holds, one population out competes its rival under strong interspecific competition, but the winner depends on which side of the stable manifold of the positive equilibrium the initial conditions lie. For this reason, the stable manifold is referred to as a separatrix.

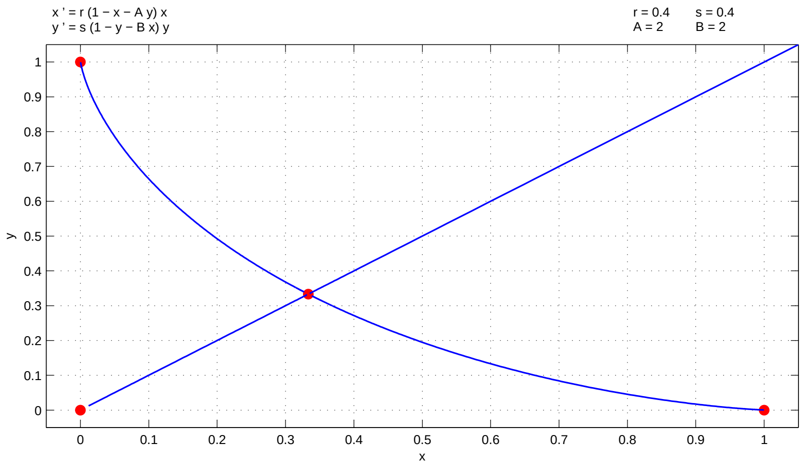

Figure 1: Stable and unstable manifold of positive equilibrium

As a second example, consider the damped pendulum equation

$$x'' + ax' + b\sin(x) = 0$$

which in vector form becomes

$$\begin{aligned} x' &= y \\ y' &= -ay - b\sin(x). \end{aligned}$$

Figure 2 depicts the phase plane. As x represents an angle, the appropriate phase space is the product of the unit circle S^1 in the plane with the real line, forming a cylinder by identifying points (x, y) with $(x + 2k\pi, y)$ (imagine rolling the plane into a scroll). There are then two distinct equilibria corresponding to the pendulum at rest hanging down ($(0, 0)$), or up ($(\pi, 0)$). The latter is a saddle point, as is easily seen from the Jacobian matrix at this point, and its stable manifold consists of two solutions which, magically, approach the up position following clockwise or counterclockwise rotation. The stable manifold of the saddle point consists of two orbits which initiate at the up position of the pendulum and, beginning clockwise or counterclockwise, approach the down position in an oscillatory manner. These invariant

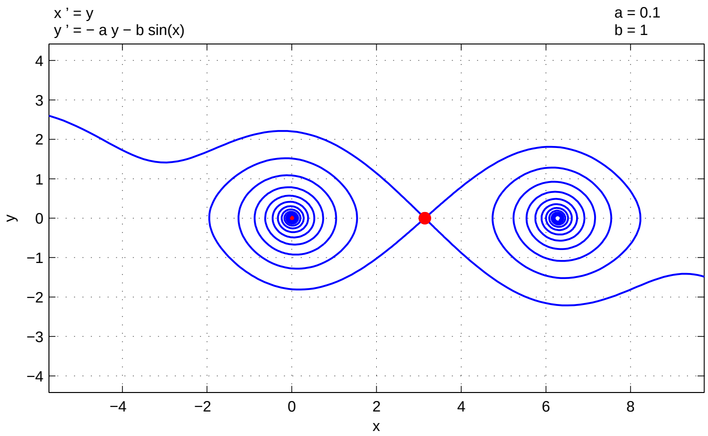

Figure 2: Stable and unstable manifold of $(\pi, 0)$

manifolds "organize" the phase diagram. The stable manifold of the saddle point spirals around the cylinder, channeling points off it into spiraling orbits asymptotic to the down position after an appropriate number of rotations. It is hard to imagine understanding this phase diagram without careful consideration of these special orbits.

The purpose of this communication is to give a simple proof of the existence of the stable and unstable manifolds corresponding to an equilibrium of saddle type for a two-dimensional system of ordinary differential equations. The proof should be accessible to students at the advanced undergraduate level. All that is required is the contraction mapping theorem and the completeness of the space of continuous functions on a bounded interval. The proof follows standard lines but is "simple" in the sense that in two dimensions, manifolds are curves and the partial differential equation for the function whose graph describes the local stable manifold reduces to an ordinary differential equation. We make no attempt to prove the most general result and are not afraid to assume unnecessary smoothness of the vector field. For a more complete treatment of the results considered here, including their extension to higher dimensions and to mappings, we recommend the texts [[1],[2],[4],[3]].

In addition to proving the existence of local stable and unstable manifolds at a saddle point equilibrium, we also establish that solutions near the saddle point behave like solutions of the linearized equations near the origin.

2 Main Results

A saddle point equilibrium z_0 of the planar system

$$z' = F(z), \quad z = (u, v)^T$$

is an equilibrium ($F(z_0) = 0$) at which the Jacobian matrix derivative

$$A = DF(z_0)$$

has real eigenvalues of opposite sign. Such points are easy to detect since they are characterized by A having a negative determinant. If we change variables by

$$w = (x, y)^T \equiv B(z - z_0)$$

where B is the matrix that diagonalizes A, that is, where

$$BAB^{-1} = \begin{pmatrix} -\lambda & 0 \\ 0 & \mu \end{pmatrix}$$

and $-\lambda < 0$ and $\mu > 0$ are the eigenvalues of A, then the system of differential equations takes the form:

$$\begin{aligned} x' &= -\lambda x + f(x, y) \\ y' &= \mu y + g(x, y). \end{aligned} \tag{1}$$

Here, f and g represent quadratic and higher order terms, that is, they and their first partial derivatives vanish at $(0, 0)$. We note that the matrix $B^{-1} = col(z_1 z_2)$ where $Az_1 = -\lambda z_1$ and $Az_2 = \mu z_2$ so that the change of variables can be written

$$z = z_0 + xz_1 + yz_2.$$

The equilibrium z_0 has been translated to the origin and x and y are components along the "stable" and "unstable" eigenvectors, respectively.

We assume that F is three times continuously differentiable so that f and g are continuous together with their partial derivatives through the third order. Taylor's theorem with remainder implies that f can be expressed as

$$f(x, y) = a(x, y)x^2 + 2b(x, y)xy + c(x, y)y^2$$

where

$$a(x, y) = \int_0^1 (1 - s) f_{xx}(sx, sy) ds$$

with identical expressions holding for b and c except that f_{xy}, respectively, f_{yy}, replace f_{xx} in the integrand. A subscript x or y indicates the partial derivative with respect to that variable. A similar expansion holds for g. Note that $a(x, y)$ and similar coefficients are continuously differentiable.

The linearization of (1) is obtained by ignoring the higher order terms f and g in (1), that is, setting $f = g = 0$. Examination of the solutions of the linearized equations suggests the kinds of results which might be expected to hold for the nonlinear system (1). First note that the coordinate axes are solution curves for the linearized system: solutions starting on the x-axis remain on the x-axis and decay to zero at the exponential rate λ as t increases to positive infinity while solutions starting on the y-axis remain there and decay to zero at the exponential rate μ as t decreases to negative infinity. We call the x-axis the stable manifold and the y-axis the unstable manifold of the linearized system. Solutions which begin off the coordinate axis lie on the hyperbolic curves defined by $(x/x_0)^{\mu}(y/y_0)^{\lambda} = 1$, where (x_0, y_0) denotes the initial value. On these curves, x tends to zero and y tends to infinity as t increases and y tends to zero and x tends to infinity as t decreases. Thus, the only solutions of the linearized system which remain near the origin as t increases (decreases) are those solutions which lie on the stable (unstable) manifolds.

We are led to expect that for the nonlinear system (1) there should be two special solutions which approach the origin as t increases forming the stable manifold. These solutions should approach the origin tangent to the x-axis and at an exponential rate. Similarly, there should be two special solutions which approach the origin as t decreases to negative infinity forming the unstable manifold. These solutions should approach the origin tangent to the y-axis at an exponential rate. Further, there should exist a neighborhood of the origin such that the only solutions of (1) which remain in this neighborhood for all positive (negative) time are those solutions starting at points of the stable (unstable) manifold. We will prove these assertions for (1).

To begin with, we look for the curve in the plane on which the two special solutions forming the stable manifold must lie. Perhaps we can more easily find that part of the curve forming the stable manifold which lies near the origin. This part of the curve we call the local stable manifold. Near the origin, this curve should be the graph of a function $y = h(x)$. Thus, we look for a twice continuously differentiable function $h(x)$ satisfying $h(0) = h'(0) = 0$ such that the graph of h,

$$G(h) = \{(x, y) : y = h(x), \quad -\epsilon < x < \epsilon\}$$

is locally invariant for (1). By this we mean that the solution $(x(t), y(t))$ belongs to $G(h)$ for the largest interval containing $t = 0$ such that $-\epsilon < x(t) < \epsilon$ provided $(x(0), y(0))$ belongs to $G(h)$. We will find such an h for which $G(h)$ is the local stable manifold for (1). If h is such that $G(h)$ is locally invariant for (1), then

$$0 = \frac{d}{dt}[y(t) - h(x(t))]$$

where $(x(t), y(t))$ is a solution of (1) satisfying $(x(0), y(0)) \in G(h)$. Differentiating the right side and evaluating at $t = 0$, using (1) and $y(0) = h(x(0))$ and setting $x = x(0)$, we obtain a differential equation for h:

$$[-\lambda x + f(x, h(x))]h'(x) = \mu h(x) + g(x, h(x)). \tag{2}$$

Below, we will show that $G(h)$ is locally invariant for (1) if (2) holds. In order that $G(h)$ is tangent to the x-axis we want $h(0) = h'(0) = 0$, so we set

$$h(x) = xu(x), u(0) = 0.$$

Putting this into (2) yields

$$xu'(x) + u(x) = \frac{\mu u(x) + x^{-1}g(x, xu(x))}{x^{-1}f(x, xu(x)) - \lambda}, \quad u(0) = 0.$$

Finally, adding $(\mu/\lambda)u(x)$ to both sides of the equation above, we obtain the equation:

$$xu'(x) + (1 + p)u(x) = xb(x, u(x)), \quad u(0) = 0, \tag{3}$$

where $p = \mu/\lambda > 0$ and

$$b(x,u) = \frac{x^{-2}[g(x,xu) + puf(x,xu)]}{x^{-1}f(x,xu) - \lambda}.$$

From this expression for b and the properties of f and g described above, it follows that b is continuously differentiable.

Observe that equation (3) has a singularity at $x = 0$ due to the factor x multiplying $u'(x)$ and therefore standard existence and uniqueness theorems fail to apply to the initial value problem. However, the singularity is mild as the reader may recall Euler's equation and the regular singular point theory usually treated in a first course in differential equations. The following simple lemma allows us to replace equation (3) by an equivalent integral equation.

Lemma 2.1 *If $q(x)$ is continuous for $x \in [-\epsilon, \epsilon]$, then the initial value problem:*

$$\begin{aligned} cxy'(x) + (1+p)y(x) &= xq(x), \quad -\epsilon \le x \le \epsilon, \\ y(0) &= 0, \end{aligned}$$

has the unique continuously differentiable solution given by

$$y(x) = \int_0^x (s/x)^{1+p} q(s)ds = x\int_0^1 s^{p+1}q(sx)ds, \quad -\epsilon \le x \le \epsilon.$$

If q is continuously differentiable, then y is twice continuously differentiable.

Proof: Multiply the equation by the absolute value of x raised to the pth power and integrate. It is easy to verify that y, given by the first integral expression above, is a continuously differentiable solution of the differential equation. To see that y' is continuously differentiable if q is, note that $y(x)/x$ is continuously differentiable from the second integral expression for y. Then the differential equation implies that y' is continuously differentiable. ∎

It follows that solving (3) is equivalent to solving the integral equation

$$u(x) = x\int_0^1 s^{1+p}b(sx, u(sx))ds. \tag{4}$$

We find a solution of (4) by applying the contraction mapping theorem.

Theorem 2.2 (Existence) *There exists* $\epsilon > 0$ *and a unique twice continuously differentiable function h, defined on* $[-\epsilon, \epsilon]$*, which satisfies* (2) *and* $h(0) = h'(0) = 0$.

Proof: It suffices to obtain a twice continuously differentiable function u satisfying (3). Fix $\delta > 0$ and $a > 0$ and let $D = \{(x, y) : |x| \leq \delta, |y| \leq a\}$, $m = \max\{|b(x, y)| : (x, y) \in D\}$ and $|b(x, y) - b(x, z)| \leq k|y - z|, (x, y), (x, z) \in D$. Choose $\epsilon < \delta$ such that $\epsilon m/(p + 2) \leq a$ and $k\epsilon/(p + 2) \leq 1/2$. Let $C = \{v : [-\epsilon, \epsilon] \rightarrow \mathbb{R} : |\mathrm{v(x)}| \leq \epsilon\mathrm{m/(p + 2), and\ v\ is\ continuous}\}$ with the norm $||v|| = \max(|v(x)| : |x| \leq \epsilon\}$. C is a complete metric space since it is a closed subset of the complete space consisting of all continuous functions on the interval. We define the mapping T on C by

$$(Tv)(x) = x \int_0^1 s^{1+p} b(sx, v(sx))ds, \quad |x| \leq \epsilon.$$

Clearly, Tv is continuous and we can estimate Tv as follows:

$$\begin{aligned} |(Tv)(x)| &\leq |x| \int_0^1 s^{1+p}|b(sx, v(sx))|ds \\ &\leq |x| \int_0^1 s^{1+p} m\, ds \leq |x|m/(p + 2) \leq m\epsilon/(p + 2). \end{aligned}$$

Hence Tv belongs to C. In order to see that T is a contraction, let $v, w \in C$. Then

$$\begin{aligned} |(Tv)(x) - (Tw)(x)| &\leq |x| \int_0^1 s^{1+p}|b(sx, v(sx)) - b(sx, w(sx))|ds \\ &\leq k\epsilon \int_0^1 s^{1+p}|v(sx) - w(sx)|ds \\ &\leq k\epsilon/(p + 2)||v - w|| \leq (1/2)||v - w||. \end{aligned}$$

So $T : C \rightarrow C$ is a contraction mapping and, as such, has a unique fixed point $u \in C$. As u is continuous, $q(x) = b(x, u(x))$ is continuous so, by Lemma 2.1, u is a continuously differentiable solution of (3). Hence q is continuously differentiable so Lemma 2.1 implies that u is twice continuously differentiable.

It follows that $h(x)$, defined by $h(x) = xu(x), |x| \leq \epsilon$, is a twice continuously differentiable solution of (2). ∎

We now return to the problem of establishing that $G(h)$ is locally invariant for (1) if (2) holds.

Theorem 2.3 (Invariance and Exponential Decay) *$G(h)$ is locally invariant for* (1)*: if $(x(t), y(t))$ is a solution of* (1) *with $(x(0), y(0)) \in G(h)$ then $(x(t), y(t)) \in G(h)$ for all t in the maximal interval containing* 0 *for which $|x(t)| < \epsilon$. Moreover, $x(t)$ satisfies the equation*

$$z' = -\lambda z + f(z, h(z)), \quad z(0) = x(0), \tag{5}$$

for t in this interval. Furthermore, by choosing $\epsilon_0 < \epsilon$ sufficiently small, $G(h) = \{(x, y) : y = h(x), -\epsilon_0 < x < \epsilon_0\}$ is positively invariant for (1)*, that is, $(x(t), y(t)) \in G(h)$ for all $t > 0$ if $(x(0), y(0)) \in G(h)$. There exists $M > 0$ such that*

$$|x(t)| + |y(t)| \leq M \exp[(-\lambda/2)t], \quad t \geq 0, \tag{6}$$

for any solution of (1) *with $(x(0), y(0)) \in G(h)$.*

Proof: Suppose that $(x(t), y(t))$ is a solution of (1) with $y(0) = h(x(0))$ and $|x(0)| < \epsilon$. We want to show that $y(t) = h(x(t))$ for all t belonging to the largest open interval containing $t = 0$ in which $|x(t)| < \epsilon$. Define by $z(t)$ the solution of (5) satisfying $z(0) = x(0)$, and let $w(t) = h(z(t))$ so that $w(0) = y(0)$. Now the reader may easily verify that if h satisfies (2) then $(z(t), w(t))$ satisfies (1). As the two solutions $(x(t), y(t))$ and $(z(t), w(t))$ of (1) satisfy the same initial conditions, they must be identically the same on the intersection of their domains by the uniqueness of solutions of (1). But this just implies that $G(h)$ is locally invariant as we claimed.

The above argument for the local invariance of $G(h)$ reveals that solutions of (1) beginning on $G(h)$ are governed by the simple scalar equation (5) so long as $|x(t)| < \epsilon$, with $y(t)$ determined by $y(t) = h(x(t))$. We now show, by choosing ϵ smaller if necessary, that $G(h)$ is positively invariant for (1). Choose ϵ_0 so small that

$$|f(x, h(x))| < (\lambda/2)|x|, \quad |x| < \epsilon_0.$$

If $(x(t), y(t))$ is a solution of (1) satisfying $y(0) = h(x(0))$, then $y(t) = h(x(t))$ so long as $|x(t)| < \epsilon_0$. For these values of t, $x(t)$ satisfies (5) and by the inequality above, it follows that $x(t)x'(t) < 0$. This implies that $|x(t)| < \epsilon_0$

and that (5) holds for all $t \geq 0$. Indeed, if $0 < x(0) < \epsilon_0$, then $0 < x(t) < \epsilon_0$ for $t > 0$ since $x = 0$ is an equilibrium point of (5). By the estimate above, it follows that

$$x'(t) < -(\lambda/2)x(t).$$

This leads immediately to $0 < x(t) < x(0)\exp[-(\lambda/2)t]$. A similar estimate holds in case that $-\epsilon_0 < x(0) < 0$. As $y(t) = h(x(t))$ and h is Lipschitz, the estimate (6) holds for some M independent of $(x(0), y(0)) \in G(h)$. This completes the proof. ∎

It is immediate from the fact that $h(0) = h'(0) = 0$ that $G(h)$ is a curve passing through $(0,0)$ tangent to the x-axis at $(0,0)$. By virtue of Theorem 2.3, $G(h)$ deserves to be called the local stable manifold of the saddle point $(0,0)$.

For small enough ϵ_0, we have shown that $G(h)$ is positively invariant for (1). $G(h)$ consists of the equilibrium point $(0,0)$ together with portions of two other orbits of (1), one with $x(0) > 0$ and one with $x(0) < 0$. Clearly, $G(h)$ is a subset of the curve obtained as the union of $(0,0)$ and the maximally extended solutions of (1) through $(-\epsilon_0/2, h(-\epsilon_0/2))$ and $(\epsilon_0/2, h(\epsilon_0/2))$. This latter curve is called the (global) stable manifold of (1). By its very definition, it is invariant for (1).

The local unstable manifold of (1) can be obtained in an identical manner as a graph of a twice continuously differentiable function $x = H(y)$ for $|y| < \epsilon$, where H satisfies $H(0) = H'(0) = 0$. For a suitably small ϵ, the graph of H will be negatively invariant for (1), that is, $(x(t), y(t))$ belongs to $G(H)$ for all $t < 0$ if $(x(0), y(0))$ belongs to $G(H)$. Moreover, if $(x(0), y(0))$ belongs to $G(H)$, then an estimate similar to (6) holds except that $(-\lambda/2)$ is replaced by $(\mu/2)$ and $t \geq 0$ is replaced by $t \leq 0$. The conclusions just described can be obtained from our previous analysis on changing t to $-t$ in (1). This introduces a minus sign on the right side of (1), transforming the stable manifold into the unstable manifold and vice versa. The global unstable manifold may be obtained exactly as for the global stable manifold.

The behavior of solutions of (1) near $(x, y) = (0,0)$ which do not belong to either the local stable or local unstable manifold can be determined by a change of variable in (1). Denote by Φ the map defined by:

$$\begin{aligned} u = x - H(y) &= \Phi_1(x, y) \\ v = y - h(x) &= \Phi_2(x, y), \end{aligned}$$

for $(x, y) \in U = \{(x, y) : |x| < \epsilon, |y| < \epsilon\}$. Observe that the Φ_i have continuous partial derivatives up to second order. As $H(0) = H'(0) = h(0) = h'(0) = 0$, the Jacobian matrix derivative of Φ satisfies

$$D\Phi(0,0) = \frac{\partial(u,v)}{\partial(x,y)}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By the inverse function theorem, Φ is an injective map of a neighborhood U' in U onto a neighborhood V of $(0,0)$ in the (u,v)-plane and its inverse, Φ^{-1}, is also twice continuously differentiable. Φ transforms (1) into a differential equation in the (u,v)-plane which may be expressed in vector form as:

$$w' = D\Phi(\Phi^{-1}(w))F(\Phi^{-1}(w)) \equiv G(w), \tag{7}$$

where $w = (u,v)$, $F(x,y) = (-\lambda x + f(x,y), \mu y + g(x,y))$ is the vector representing the right side of (1) and $D\Phi$ is the Jacobian derivative of F evaluated at the point $\Phi^{-1}(w)$. The expression for the vector function G implies that G is continuously differentiable in the neighborhood $V = \Phi(U')$ of $(0,0)$. Φ maps orbits of (1) in U' onto orbits of (7) in V while Φ^{-1} maps orbits of (7) in V onto orbits of (1) in U. The reader should verify that (7) can be expressed in component form as:

$$\begin{aligned} u' &= -\lambda u + \bar{f}(u,v) \\ v' &= \mu v + \bar{g}(u,v), \end{aligned}$$

where $\bar{f}$ and $\bar{g}$ and their partial derivatives of the first order exist and are continuous in V and vanish at $(0,0)$.

The system (7) has the same formal appearance as (1) but it has a property which makes the analysis of its small solutions considerably easier than would be the case for (1). The transformation Φ maps the local stable manifold ($y = h(x)$) of (1) to the u-axis, i.e., $v = 0$, and it maps the local unstable manifold ($x = H(y)$) of (1) to the v-axis, i.e., to $u = 0$. As the local stable and unstable manifolds consist of solution curves for (1), their images are solution curves for (7). Consequently, these axes are also the nullclines for the system (7), i.e., $v' = 0$ on the u-axis and $u' = 0$ on the v-axis, or

$$\bar{f}(0,v) = \bar{g}(u,0) = 0$$

for all small u and v. Furthermore, the implicit function theorem, applied to the equation $u' = 0$ ($v' = 0$) implies that in some neighborhood of the

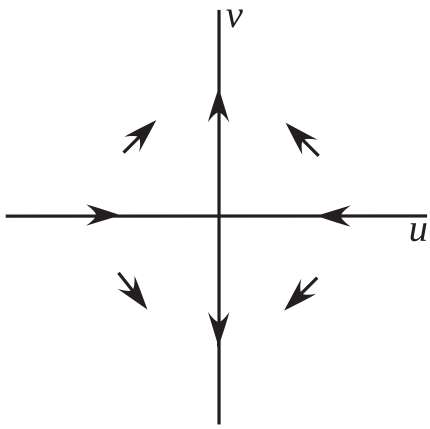

Figure 3: Direction Field for (7)

origin, $u' = 0$ ($v' = 0$) only on the v-axis (u-axis). The fact that under the transformation Φ the local stable and unstable manifolds coincide with the nullclines of (7) greatly simplifies the analysis of (7), in particular because it implies that each open quadrant of the $u - v$ plane is invariant for (7). By this we mean that a solution remains in the open quadrant in which it begins. In Figure 1, the qualitative nature of the vector field (7) is depicted. From a qualitative viewpoint, Figure 1 could just as well describe the linear system obtained from (1) or (7) by neglecting f and g, except of course that the figure is valid only near $(0, 0)$.

The behavior of solutions, in a neighborhood of the origin, beginning in each open quadrant of the $u-v$ plane can be analyzed separately. Consider a solution beginning near the origin in the open first quadrant of the $u-v$ plane. From the direction of the flow of (7) on the axes, we infer that $u' < 0$ and $v' > 0$ in the open first quadrant. It follows immediately that our solution is such that each component is monotone in the time variable and hence it must either approach the boundary of the domain V or approach an equilibrium in V, in either case remaining in the first quadrant, as t increases. The latter alternative is impossible since the origin is the only equilibrium in V and it can't be approached by our solution since $v(t) > v(0) > 0$ for $t > 0$. Hence, every solution beginning in the open first quadrant remains in the open first quadrant and approaches the boundary of V as t increases. Similar analysis applies to solutions beginning in each of the other quadrants: all solutions beginning in the interior of one of the quadrants must approach the boundary of V as t increases. Similar conclusions hold on reversing the direction of time. We conclude that a solution of (7) which remains in a compact neighborhood of the origin in V for all $t > 0$ must lie on the local stable manifold ($v = 0$) and approach the origin at an exponential rate as t tends to infinity. Similarly, a solution which remains in a compact neighborhood of the origin in V for all $t < 0$ must lie on the unstable manifold ($u = 0$) and approach the origin as t tends to $-\infty$ at an exponential rate. These conclusions carry over to (1) under the transformation Φ^{-1}. We state these conclusions formally:

Theorem 2.4 (Behavior of Small Solutions) *There exists a neighborhood, W, of the origin such that any solution $(x(t), y(t))$ of (1) which remains in W for all $t \geq 0$ must belong to the local stable manifold $G(h)$. Similarly, any solution which remains in W for all $t \leq 0$ must belong to the local unstable manifold $G(H)$.*

References

[1] J. K. Hale, *Ordinary Differential Equations*, R. E. Krieger Publishing Co., Malabar, Florida, 1980.

[2] M.W.Hirsch, C. Pugh, M. Shub, *Invariant Manifolds* Lecture Notes in Mathematics, 583, Springer-Verlag, New York, 1976.

[3] W. Melo and J.Palis, *Geomeric Theory of Dynamical Systems*, Springer-Verlag, New York, 1982.

[4] M. Shub, *Global Stability of Dynamical Systems*, Springer-Verlag, New York, 1987.

[5] P. Waltman, *A Second Course in Elementary Differential Equations*, Academic Press, New York, 1986.