

Robust Persistence for Semidynamical Systems

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Abstract

Using properties of internally chain transitive sets, we show that uniform persistence of a semi-dynamical system is robust to small perturbations of the semi-dynamical system under mild compactness conditions.

Key words: Chain transitive sets; uniform persistence.

1 Introduction

The notion of chain recurrence, introduced by Conley [1], is a way of getting at the recurrence properties of a dynamical system. It has remarkable connections to the structure of attractors. Chain recurrence has been used to characterize the property of uniform persistence (or permanence) for dynamical systems, an idea that arose out of population biology; see Garay [4], Hofbauer and So [8], Schreiber [10], Smith and Zhao [12] and Hirsch, Smith and Zhao [7]. Looked at abstractly, uniform persistence is the notion that a closed subset of the state space (e.g., the set of extinction for one or more populations) is repelling for the dynamics on the complementary set. Following our work in [7] for maps, we show here that uniform persistence is robust under a broad class of perturbations in the dynamics. The robustness of uniform persistence is useful in establishing the robustness of global asymptotically stability of an equilibrium solution (see [11]).

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We take as our setting here a continuous-time semiflow on a metric space. See [7] for the case of a discrete-time semiflow generated by a non-invertible map.

2 Chain transitive sets

Let X be a metric space with metric d and let $\Phi_t : X \rightarrow X, t \in [0, \infty)$ be a continuous semiflow on X . That is, $(x, t) \rightarrow \Phi_t(x)$ is continuous, $\Phi_0 = id_X$ and $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for $t, s \geq 0$. A subset A of X is said to be positively invariant for Φ if $\Phi_t(A) \subset A$ for all $t \geq 0$ and **invariant** if $\Phi_t(A) = A$ for all $t \geq 0$. The **omega limit set** of A is defined by $\omega(A) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \Phi_s(A)}$. The special case $A = \{x\}$ will be denoted $\omega(x)$.

A subset $A \subset X$ is said to be an **attractor** for Φ if A is nonempty, compact and invariant, and there exists some open neighborhood U of A in X such that $\omega(U) = A$. Set A is a **global attractor** for Φ if it is an attractor for which $\omega(x)$ is non-empty and $\omega(x) \subset A$ for all $x \in X$. For a nonempty invariant set M , the set $W^s(M) := \{x \in X : \lim_{t \rightarrow \infty} d(\Phi_t(x), M) = 0\}$ is called the **stable set** of M . A continuous function $\phi : (-\infty, \infty) \rightarrow X$ is called a **globally defined solution** of Φ through $x = \phi(0)$ if $\Phi_t(\phi(s)) = \phi(t + s)$ for all $t \geq 0$ and all real s . There may be no full solution through x and even if there is one, it may not be unique. Of course, a point of an invariant set always has at least one full solution whose range is contained in the invariant set. For a given full solution ϕ we define its **alpha limit set** as $\alpha(\phi) = \bigcap_{t \leq 0} \overline{\bigcup_{s \leq t} \phi(s)}$.

A nonempty invariant set $A \subset X$ for Φ_t is said to be **internally chain transitive** if for any $a, b \in A$ and any $\epsilon > 0, t_0 > 0$, there is a finite sequence $\{x_1 = a, x_2, \dots, x_m, x_{m+1} = b; t_1, \dots, t_m\}$ with $x_i \in A$ and $t_i \geq t_0, 1 \leq i \leq m$, such that $d(\Phi_{t_i}(x_i), x_{i+1}) < \epsilon$ for all $1 \leq i \leq m$. The sequence $\{x_1, \dots, x_{m+1}; t_1, \dots, t_m\}$ is called an (ϵ, t_0) -chain in A connecting a and b . A is said to be **internally chain recurrent** if for every $a \in A, t_0 > 0$, and $\epsilon > 0$ there exists an (ϵ, t_0) -chain in A connecting a to itself. We usually drop the adjective "internally" and simply say that A is chain transitive or chain recurrent. Obviously, chain transitivity implies chain recurrence; if A is connected and chain recurrent then it is chain transitive (see [1,9]).

The most important examples of chain transitive sets are limit sets.

Lemma 1 *Let Φ be a semiflow on X . Then the omega limit set of any pre-compact positive orbit is chain transitive. The same holds for the alpha limit set of any full solution ϕ for which $\{\phi(t) : t \leq 0\}$ is compact.*

Conley [1] proved Lemma 1 for flows on a compact set. Lemma 1 is proved in [7] for semiflows; the first assertion is also proved in [9]. Other examples of chain transitive sets include omega limit sets of pre-compact orbits of asymptotically autonomous semiflows (see [9]).

Let A and B be two nonempty compact subsets of X . Recall that the Hausdorff distance between A and B is defined by

$$d_H(A, B) := \max(\sup\{d(x, B) : x \in A\}, \sup\{d(x, A) : x \in B\}).$$

The following result, proved in [7] for maps, says that limits of chain transitive sets are chain transitive.

Lemma 2 *Let Φ and Φ^n be semiflows on X for $n \geq 1$. Let $\{D_n\}$ be a sequence of nonempty compact subsets of X with $\lim_{n \rightarrow \infty} d_H(D_n, D) = 0$ for some compact subset D of X . Assume that for each $n \geq 1$, D_n is invariant and chain transitive (chain recurrent) for Φ^n . If for each $T > 0$, $\Phi^n \rightarrow \Phi$ uniformly for $(x, t) \in [D \cup (\cup_{n \geq 1} D_n)] \times [0, T]$, then D is invariant and chain transitive (chain recurrent) for Φ .*

Proof. It is easy to see that $K = D \cup (\cup_{n \geq 1} D_n)$ is compact and D is invariant for Φ . By uniform continuity and uniform convergence, for any $\epsilon > 0$ and $t_0 > 0$ there exists $\delta \in (0, \epsilon/3)$ and a natural number N such that for $n \geq N$, $t \in [0, 2t_0]$ and $u, v \in K$ with $d(u, v) < \delta$, we have $d(\Phi_t^n(u), \Phi_t(v)) \leq d(\Phi_t^n(u), \Phi_t(u)) + d(\Phi_t(u), \Phi_t(v)) < \epsilon/3$. Fix $n > N$ such that $d_H(D_n, D) < \delta$. For any $a, b \in D$, there are points $x, y \in D_n$ such that $d(x, a) < \delta$ and $d(y, b) < \delta$. As D_n is chain transitive for Φ^n there is a (δ, t_0) -chain $\{z_1 = x, z_2, \dots, z_{m+1} = y; t_1, \dots, t_m\}$ in D_n for Φ^n , with $t_0 \leq t_i < 2t_0$ connecting x to y . For each $i = 2, \dots, m$ we can find $w_i \in D$ with $d(w_i, z_i) < \delta$ since D_n is contained in the δ -neighborhood of D . Let $w_1 = a, w_{m+1} = b$. We then have

$$\begin{aligned} d(\Phi_{t_i}(w_i), w_{i+1}) &\leq d(\Phi_{t_i}(w_i), \Phi_{t_i}^n(z_i)) + d(\Phi_{t_i}^n(z_i), z_{i+1}) + d(z_{i+1}, w_{i+1}) \\ &< \epsilon/3 + \delta + \delta < \epsilon \end{aligned}$$

for $i = 1, \dots, m$. Thus $\{w_1 = a, w_2, \dots, w_{m+1} = b; t_1, \dots, t_m\}$ is an (ϵ, t_0) -chain for Φ in D connecting a to b . ■

We observe that if in Lemma 2 D_n is an omega limit set for Φ_n (and therefore chain transitive by Lemma 1), the set D need not be an omega limit set for the limit semiflow Φ , although it must be chain transitive. Easy examples are

constructed with $\Phi^n = \Phi$. For example, consider the flow generated by the planar vector field given in polar coordinates by

$$r' = 0, \quad \theta' = 1 - r.$$

The unit circle $D = \{r = 1\}$, consisting of equilibria, is chain transitive but is not an omega limit set of any point, yet D is the Hausdorff limit of the omega limit sets $D_n = \{r = 1 + \frac{1}{n}\}$.

3 Robustness of uniform persistence

Throughout this section, X is a metric space and Φ is a semiflow on X . Let $X_0 \subset X$ be an open set and $\partial X_0 = X \setminus X_0$. Define $M_\partial = \{x \in \partial X_0 : \Phi_t(x) \in \partial X_0, t \geq 0\}$, which may be empty. Note that ∂X_0 need not be the boundary of X_0 as the notation suggests and neither X_0 nor ∂X_0 are assumed to be positively invariant. This peculiar notation has become standard in persistence theory (see, e.g., [15]). We assume hereafter that every positive orbit of Φ is pre-compact. Motivated by ideas in [16], a continuous function $p : X \rightarrow [0, \infty)$ satisfying condition:

(P) $p(\Phi_t(x)) > 0$ for $t > 0$ if either $p(x) = 0$ and $x \in X_0$ or if $p(x) > 0$,

will be called a generalized distance function for Φ . An important example is

$$p(x) \equiv d(x, \partial X_0) \tag{1}$$

in case X_0 is positively invariant.

Φ is said to be **uniformly persistent** with respect to $(X_0, \partial X_0, p)$ if there exists $\eta > 0$ such that

$$\liminf_{t \rightarrow \infty} p(\Phi_t(x)) \geq \eta$$

for all $x \in X_0$.

The no-cycle condition is a central assumption in persistence theory. The necessary definitions follow. Let A and B be two isolated invariant sets. A is said to be **chained** to B , written $A \rightarrow B$, if there exists a globally defined solution ϕ through some $x \notin A \cup B$ whose range has compact closure and such that $\omega(x) \subset B$ and $\alpha(\phi) \subset A$. A finite sequence $\{M_1, \dots, M_k\}$ of isolated invariant sets is called a **chain** if $M_1 \rightarrow M_2 \rightarrow \dots \rightarrow M_k$. The chain is called a **cycle** if $M_k = M_1$.

The following result is Theorem 4.3 in [7] in the special case where p is given by (1). The proof there is easily extended to include general p .

Theorem 3 *Let p be a generalized distance function for semiflow Φ . Assume that*

(C1) Φ has a global attractor A .

(C2) *There exists a finite sequence $M = \{M_1, \dots, M_k\}$ of pairwise disjoint, compact and isolated invariant sets in ∂X_0 with the following properties:*

- $\cup_{x \in M_\theta} \omega(x) \subset \cup_{i=1}^k M_i$,
- no subset of M forms a cycle in ∂X_0 ,
- M_i is isolated in X ,
- $W^s(M_i) \cap p^{-1}(0, \infty) = \emptyset$ for each $1 \leq i \leq k$.

Then there exists $\delta > 0$ such that for any compact chain transitive set L with $L \not\subset M_i$ for all $1 \leq i \leq k$, there holds $\min_{x \in L} p(x) > \delta$.

Sketch of modifications in the proof of Theorem 4.3 [7]: We refer the reader to that proof and its notation, which will be followed in this brief sketch (in particular, the dynamics is generated by a mapping $f : X \rightarrow X$ and (P) is modified in the obvious fashion). The first claim is that there exists $\epsilon > 0$ such that $\sup\{p(x) : x \in L\} > \epsilon$ holds for all chain transitive sets L not contained in any M_i . Arguing by contradiction as in the original proof we arrive at chain transitive set D (limit of sets D_n) satisfying $p(x) = 0$ for all $x \in D$. If $x \in D \cap X_0$ then (P) implies $p(f(x)) > 0$, a contradiction to $f(x) \in D$, so we conclude that $D \subset \partial X_0$. The remainder of the proof of the claim is unchanged.

The second part of the proof begins by contradicting the conclusion of the result, obtaining chain transitive set L , with L not contained in any M_i , as a limit of chain transitive sets L_n , each not contained in any M_i , and with $\lim_{n \rightarrow \infty} \inf\{p(x) : x \in L_n\} = 0$. So we find $x_n \in L_n$ with $p(x_n) \rightarrow 0$, implying that L contains a point a with $p(a) = 0$. By the claim, we can find point $b \in L$ such that $p(b) > \epsilon$. At this point the proof continues as in Theorem 4.3 [7] with the construction of an asymptotic pseudo-orbit except that $d(y, \partial X_0)$ is replaced in each occurrence by $p(y)$. We conclude that the subsequential limit $x \in L$ of the pseudo-orbit satisfies $p(x) \geq \epsilon$, but this doesn't imply $x \in X_0$. Furthermore, as we do not assume that X_0 is positively invariant, the argument that $m_j - l_j$ is unbounded can be modified as follows: $a = f^m(x)$ contradicts that $p(a) = 0 = p(f^m(x))$, $p(x) \geq \epsilon$, and condition (P) which requires $p(f^m(x)) > 0$. Continuing as in the original argument, we arrive at $p(f^n(x)) \leq \epsilon$ for $n \geq 1$. Thus, $p(y) \leq \epsilon$ on the chain transitive set $\omega(x) \in L$. But this contradicts (C2) as in the original proof. ■

In the special case that $L = \omega(x)$ for $x \in X_0$, the condition $L \subset M_i$ for some i violates the last hypothesis of (C2), so we conclude that

$$\min_{y \in \omega(x)} p(y) > \delta, \quad x \in X_0.$$

This is precisely uniform persistence. Therefore, Theorem 3 includes a uniform persistence result. In case p is given by (1), this result is well-known. See [3,8] for maps and [6] for semiflows. Indeed, the assumption (C1) can be replaced by some weaker compactness assumptions near ∂X_0 , see, e.g., [15] for a detailed discussion in the context of continuous semiflows.

The no cycle condition (C2) may be equivalently formulated using a Morse decomposition of the maximal compact invariant set for the restriction of Φ to ∂X_0 . See [7].

Theorem 4 *Let Φ^m , $m \geq 0$, be a sequence of semiflows on X such that every positive orbit for Φ^m has compact closure. Let $\omega_m(x)$ denote the omega limit of x for Φ^m , and set $W = \overline{\cup_{m \geq 0, x \in X} \omega_m(x)}$. Assume W is compact and for each $T > 0$, $\Phi^m \rightarrow \Phi^0$ uniformly for $(x, t) \in W \times [0, T]$. In addition, assume:*

- (A1) Φ^0 satisfies (C1) and (C2) of Theorem 3 with generalized distance function p for Φ^0 .
- (A2) there exist $\eta_0 > 0$ and a positive integer N_0 such that for $m \geq N_0$ and $x \in X_0$, $\limsup_{t \rightarrow \infty} d(\Phi_t^m(x), M_i) \geq \eta_0$, $1 \leq i \leq k$.

Then there exist $\eta > 0$ and a positive integer N such that $\liminf_{t \rightarrow \infty} p(\Phi_t^m(x)) \geq \eta$ for $m \geq N$ and $x \in X_0$.

Proof. Assume that, by contradiction, there exists a sequence $\{x_k\}$ in X_0 and positive integers $m_k \rightarrow \infty$ satisfying $\liminf_{t \rightarrow \infty} p(\Phi_t^{m_k}(x_k)) \rightarrow 0$ as $k \rightarrow \infty$. By Lemma 1, $\omega_{m_k}(x_k)$ is a compact chain transitive set for Φ^{m_k} . In the compact metric space of all compact subsets of W with Hausdorff distance d_H , the sequence $\{\omega_{m_k}(x_k)\}$ has a convergent subsequence. Without loss of generality, we assume that for some nonempty compact $L \subset W$, $\lim_{k \rightarrow \infty} d_H(\omega_{m_k}(x_k), L) = 0$. Clearly, there exist $y_k \in \omega_{m_k}(x_k)$ such that $\lim_{k \rightarrow \infty} p(y_k) = 0$, and hence $L \cap p^{-1}(0) \neq \emptyset$. By Lemma 2, L is chain transitive for Φ^0 . Since $L \cap p^{-1}(0) \neq \emptyset$, Theorem 3, applied to Φ^0 , implies $L \subset M_i$ for some i . But $\omega_{m_k}(x_k) \rightarrow L$ gives a contradiction to assumption (A2). ■

Theorem 5 (Uniform persistence uniform in parameters) *Let Λ be a metric space with metric ρ . For each $\lambda \in \Lambda$, let Φ^λ be a semiflow on X such that $\Phi_t^\lambda(x)$ is continuous in (λ, x, t) . Assume that every positive orbit for Φ^λ has compact closure in X , and that $\bigcup_{\lambda \in \Lambda, x \in X} \omega_\lambda(x)$ has compact closure, where $\omega_\lambda(x)$ denotes the omega limit of x for semiflow Φ^λ . Let $\lambda_0 \in \Lambda$ be fixed, and assume further that*

(B1) Φ^{λ_0} satisfies (C1) and (C2) of Theorem 3 with generalized distance function p for Φ^{λ_0} .

(B2) There exists $\delta_0 > 0$ such that for any $\lambda \in \Lambda$ with $\rho(\lambda, \lambda_0) < \delta_0$ and any $x \in X_0$, $\limsup_{t \rightarrow \infty} d(\Phi_t^\lambda(x), M_i) \geq \delta_0$, $1 \leq i \leq k$.

Then there exists $\delta > 0$ such that $\liminf_{n \rightarrow \infty} p(\Phi_t^\lambda(x)) \geq \delta$ for any $\lambda \in \Lambda$ with $\rho(\lambda, \lambda_0) < \delta$ and any $x \in X_0$.

Proof. Clearly, (B1) implies that (A1) holds for $\Phi^0 := \Phi^{\lambda_0}$. If the conclusion were false we could find sequences $x_k \in X_0$ and λ_k with $\lambda_k \rightarrow \lambda_0$ such that $\liminf_{t \rightarrow \infty} p(\Phi_t^k(x_k)) \rightarrow 0$ as $k \rightarrow \infty$, where $\Phi^k := \Phi^{\lambda_k} \rightarrow \Phi^0$ uniformly on $W \times [0, T]$ for each $T > 0$ by uniform continuity of $(\lambda, x, t) \rightarrow \Phi_t^\lambda(x)$ on compact (λ, x, t) -sets. But this contradicts Theorem 4. ■

Theorem 5 is very similar to [12, Theorem 4.3]. The difference lies in that the existence of a global attractor $A_0 \subset X_0$ for $\Phi^{\lambda_0} : X_0 \rightarrow X_0$ is assumed in [12, Theorem 4.3].

4 An application

A microbial population growing on a substrate in a continuously-stirred tank reactor occupying an open and connected subset of R^N with smooth boundary, can be modeled by a system of reaction diffusion equations for microbial biomass density u and substrate concentration S . The equations derived in [2] are

$$\begin{aligned} S_t &= d_0 \Delta S - \gamma^{-1} u f(S) \\ u_t &= d \Delta u + u[f(S) - k], \quad x \in \Omega, \quad t > 0 \end{aligned} \tag{2}$$

with Robin boundary conditions

$$\begin{aligned}
S^0(x) &= \frac{\partial S}{\partial \nu}(t, x) + r_0(x)S(t, x) \\
0 &= \frac{\partial u}{\partial \nu}(t, x) + r(x)u(t, x), \quad x \in \partial\Omega, \quad t > 0
\end{aligned} \tag{3}$$

and initial conditions

$$\begin{aligned}
S(0, x) &= S_0(x) \\
u(0, x) &= u_0(x), \quad x \in \Omega.
\end{aligned} \tag{4}$$

Here, diffusivities $d, d_0 > 0$, yield constant $\gamma > 0$, $S^0, r, r_0, k \geq 0$, ν is the outward normal to the boundary $\partial\Omega$, and S^0, r, r_0 are continuous and do not vanish identically on their respective domains.

The nutrient uptake rate $f : R_+ \rightarrow R_+$ is assumed to be continuously differentiable and to satisfy $f(0) = 0$. The Monod function given by

$$f(S) = \frac{mS}{a + S}$$

is often used.

It is shown in [2] that equations (2)-(4) generate a semiflow on the space $X \equiv C_+(\overline{\Omega})^2$ given by $\Phi_t(S_0, u_0) = (S(t, \bullet), u(t, \bullet))$, $t > 0$. Semiflow Φ satisfies $\Phi_t(S_0, 0) = (S(t, \bullet), 0) \rightarrow (S_*, 0)$ as $t \rightarrow \infty$, uniformly for S_0 in bounded subsets of $C_+(\overline{\Omega})$, where S_* is the solution of the linear boundary value problem obtained from equations (2)-(3) by setting $S_t, u = 0$. Finally, we note that Φ_t is completely continuous for each $t > 0$, orbits of bounded sets are bounded, and Φ has a global attractor in X .

The stability of the washout equilibrium $(S_*, 0)$ is determined by the eigenvalue problem

$$\begin{aligned}
\lambda v &= d\Delta v + v[f(S_*) - k], \quad x \in \Omega \\
0 &= \frac{\partial v}{\partial \nu}(x) + r(x)v(x), \quad x \in \partial\Omega
\end{aligned} \tag{5}$$

If the largest eigenvalue is positive, then $(S_*, 0)$ is unstable, if it is negative, then $(S_*, 0)$ is locally asymptotically stable.

Let $X_0 = \{(S_0, u_0) \in X : u_0 \neq 0\}$. An easy maximum principle argument [2] establishes that if $(S_0, u_0) \in X_0$, then $u(t, x) > 0$ for all $x \in \overline{\Omega}$ and $t > 0$. Therefore, X_0 is positively invariant for Φ . The complementary set $\partial X_0 = \{(S_0, u_0) \in X_+ : u_0 = 0\}$ is closed and positively invariant. The

function $p : X \rightarrow [0, \infty)$, defined by

$$p(S_0, u_0) \equiv \min_{x \in \bar{\Omega}} u_0(x),$$

is continuous and, by the arguments above, satisfies the condition $p(\Phi_t(S_0, u_0)) > 0$ for $t > 0$ if either $p(S_0, u_0) = 0$ and $(S_0, u_0) \in X_0$ or if $p(S_0, u_0) > 0$. Thus, p is a generalized distance function for Φ .

Let $M > 0$ and define the set Λ as:

$$\{(k, S^0, f) \in R_+ \times C_+(\partial\Omega) \times C^1(R_+, R_+) : |S^0(x)| \leq M, f(0) = 0, f(S) \leq MS\}$$

where the inequalities hold for all values of the indicated quantities. We metrize Λ with the product metric, denoted by ρ , using the usual uniform norm on the second factor, and a metric providing uniform convergence on compact subsets of R_+ on the third factor (e.g. $d(f, g) = \sum_n 2^{-n} \frac{q_n(f, g)}{1 + q_n(f, g)}$ where $q_n(f, g) = \sup_{x \in [0, n]} |f(x) - g(x)|$.)

Our main application of Theorem 5 is that the bacterial population is uniformly persistent if the largest eigenvalue is positive.

Theorem 6 *Suppose the largest eigenvalue of (5) is positive for some $\lambda_0 = (k, S^0, f) \in \Lambda$. Then there exists $\epsilon, \delta > 0$ such that for all $\lambda \in \Lambda$ with $\rho(\lambda_0, \lambda) < \delta$ and all $(S_0, u_0) \in X_0$, we have*

$$\liminf_{t \rightarrow \infty} u^\lambda(t, x) > \epsilon, \quad x \in \bar{\Omega}.$$

Proof. We first note that the joint continuity of the map $(\lambda, S_0, u_0, t) \rightarrow (S^\lambda(t, \bullet), u^\lambda(t, \bullet))$, uniformly on bounded t sets, is a standard result which we do not address here.

The bounds defining the parameter set Λ ensure that hypotheses (F1)-(F3) in section 2 of [2] hold with a common set of bounds in (F2),(F3) uniformly in $(k, S^0, f) \in \Lambda$. Therefore, estimates obtained in Thm. 2.5, Cor.2.6, and Thm. 2.7 of [2] hold uniformly in Λ . We conclude that $\cup_{\lambda \in \Lambda, x \in X} \omega_\lambda(x)$ has compact closure in X and Φ^{λ_0} has a global attractor.

Since $\omega_{\lambda_0}(S_0, 0) = (S_*^{\lambda_0}, 0)$ for all $(S_0, 0) \in \partial X_0$, an acyclic covering consisting of the single equilibrium $M = \{(S_*^{\lambda_0}, 0)\}$ exists. One must only check that there is no globally defined solution in ∂X_0 homoclinic to equilibrium M . The orbit of such a solution would be a bounded invariant set $N \subset \partial X_0$ with $N \neq M$ so, as already noted above, due to the affine nature of Φ^{λ_0} restricted to ∂X_0 , $\bar{d}(\Phi_t^{\lambda_0}(N), M) := \sup_{x \in \Phi_t^{\lambda_0}(N)} d(x, M) \rightarrow 0$ as $t \rightarrow \infty$. This contradicts $\Phi_t^{\lambda_0}(N) = N \neq M$, proving acyclicity.

The isolatedness of M in X will follow from the verification of (B2) below. That $W^s(M)$ contains no point (S_0, u_0) with $p(S_0, u_0) > 0$ also follows from (B2) because $(S_0, u_0) \in X_0$.

We now verify hypothesis (B2) of Theorem 5. If it fails to hold, we could find a sequence $x_n = (S_0^n, u_0^n) \in X_0$ and $\lambda_n = (k_n, S_n^0, f_n) \rightarrow (k, S^0, f)$ with the property that

$$\limsup_{t \rightarrow \infty} [\|S^{\lambda_n}(t, \bullet) - S_*^{\lambda_0}(\bullet)\| + \|u^{\lambda_n}(t, \bullet)\|] \rightarrow 0. \quad (6)$$

By shifting the time axis, we may as well replace $\limsup_{t \rightarrow \infty}$ by $\sup_{t \geq 0}$ in the above equation and assume that $u_0^n(x) > 0$ for all $x \in \bar{\Omega}$. Consequently, given $\eta > 0$, for all large n , $f_n(S^{\lambda_n}(t, x)) - k_n > f(S_*^{\lambda_0}(x)) - k - \eta$ for $t \geq 0$ and $x \in \Omega$. Thus, $v^n \equiv u^{\lambda_n}$ satisfies

$$v_t^n \geq d\Delta v^n + v_n[f(S_*^{\lambda_0}(x)) - k - \eta]$$

with corresponding boundary conditions. Our assumptions concerning the largest eigenvalue of (5) together with the continuity of the largest eigenvalue to perturbations of the potential $f(S_*) - k$ imply that for sufficiently small $\eta > 0$, eigenvalue problem (5) with potential $f(S_*^{\lambda_0}(x)) - k - \eta$ has a positive eigenvalue μ and a positive eigenfunction $v(x)$. For each n we can find $c_n > 0$ such that $u_0^n(x) > c_n v(x)$ for $x \in \Omega$, and a standard comparison result implies that for large n

$$v^n(t, x) \geq c_n e^{\mu t} v(x), \quad x \in \Omega, t > 0,$$

a contradiction to the boundedness of orbits and to (6) above. This proves that (B2) holds. ■

We remark that the hypotheses of Theorem 6 imply the existence of an equilibrium solution $(\hat{S}^{\lambda_0}(x), \hat{u}^{\lambda_0}(x))$ satisfying $0 < \hat{u}^{\lambda_0}(x)$ and $0 < \hat{S}^{\lambda_0}(x) < S_*^{\lambda_0}(x)$ for the reference parameter λ_0 . See [2] or apply general results in [17]. Furthermore, the washout steady state is a global attractor if the largest eigenvalue is negative and f is monotone increasing. See Theorem 1.3 of [2].

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