

Monotone Maps: a review

M.W. Hirsch* Hal Smith †

September 3, 2004

This paper is dedicated to Jim Cushing on the occasion of his 62th birthday

Abstract

The aim of this paper is to provide a brief review of the main results in the theory of discrete-time monotone dynamics.

Keywords: Monotone maps, strongly monotone maps, strongly order preserving maps.

1 Introduction and Motivating Examples

A map $T : X \rightarrow X$ is monotone if (X, \leq) is a partially ordered set and $x, y \in X$, $x \leq y \implies Tx \leq Ty$. Typically, X will be a subset of a Banach space Y with a cone Y_+ of positive elements and $x \leq y$ is equivalent to $y - x \in Y_+$.

From a historical point of view, one of the principle motivations for the study of monotone maps is their importance in the study of periodic solutions to periodic quasimonotone systems of differential equations. The monograph of Krasnoselskii [43] is a pioneering work in this direction.

To fix ideas, consider the celebrated periodic Lotka-Volterra competition equations studied by de Mottoni and Schiaffino [20]

$$\begin{aligned}x' &= x[r(t) - a(t)x - b(t)y] \\y' &= y[s(t) - c(t)x - d(t)y]\end{aligned}\tag{1}$$

where r, s, a, b, c, d are periodic of period one and $a, b, c, d \geq 0$. The period map $T : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$, defined by

$$(x(0), y(0)) \rightarrow (x(1), y(1)).$$

Its fixed points (periodic points) are in one-to-one correspondence with the periodic (subharmonic) solutions of (1). If K denotes the closed fourth quadrant of \mathbb{R}^2 , then

*Department of Mathematics, University of California, Berkeley, CA 94720-3840. Supported by NSF Grant DMS 9700910

†Department of Mathematics, Arizona State University, Tempe, AZ 85287-1804. Supported by NSF Grant DMS 0107160

K is a cone in \mathbb{R}^2 generating the partial order $\leq_K: p \leq q \iff p_1 \leq q_1, p_2 \geq q_2$. It is well-known that T is a monotone map respecting this order. Moreover, T has the important property, not shared with general monotone maps, that it is an orientation-preserving homeomorphism.

The work of de Mottoni and Schiaffino [20] inspired a great many other authors. See Hale and Somolinos [26], Smith [73, 74], Liang and Jiang [55], and Wang and Jiang [92, 93, 94]. Cushing [12, 13] uses global bifurcation methods to study the bifurcation and stability of coexistence periodic orbits from single-population periodic orbits.

In a similar way, periodic solutions for systems of second order parabolic partial differential equations with time-periodic data can be analyzed by considering period maps in appropriate function spaces. Here monotonicity comes from classical maximum principles. Hess [28] remains an up-to-date survey. See also Alikakos *et al.* [3], Zhao [100] and Poláčik [65]. The paper of Hess and Lazer [29] initiates the development of an abstract theory of two-species competition, motivated by the time-periodic reaction diffusion model:

$$\begin{aligned} u_t &= k_1 \Delta u + u[a - bu - cv], \quad x \in \Omega \\ v_t &= k_2 \Delta v + v[d - eu - fv] \\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega \end{aligned} \tag{2}$$

The coefficients $a = a(x, t) = a(x, t + \tau), \dots, f$ are periodic of common period τ . System (2) gives rise to a period map $T : Y_+ \times Y_+ \rightarrow Y_+ \times Y_+$, where Y_+ is the cone of nonnegative functions in some Banach space Y , given by

$$(u(\cdot, 0), v(\cdot, 0)) \rightarrow (u(\cdot, \tau), v(\cdot, \tau)).$$

The work of Hess and Lazer has inspired a very large amount of work on competitive dynamics. See Hale and Somolinos [26], Smith [73, 74], Hess and Lazer [29], Hutson *et al.* [35], Hsu *et al.* [36], Smith and Thieme [79], Takáč [87], Wang and Jiang [93, 94, 92], Liang and Jiang [55], Zanolin [98].

Remarkable results are known for scalar periodic parabolic equations on a compact interval with standard boundary conditions. Chen and Matano [9] show that every forward (backward) bounded solution is asymptotic to a periodic solution; Brunovsky *et al.* [8] extend the result to more general equations. Chen *et al.* [10] give conditions for the period map to generate Morse-Smale dynamics and thus be structurally stable. Although monotonicity of the period map is an important consideration in these results, it is not the key tool. The fact that the number of zeros on the spatial interval of a solution of the linearized equation is non-increasing in time is far more important. See Hale [25] for a nice survey.

A different theme in order-preserving dynamics originates in the venerable subject of nonlinear elliptic and parabolic boundary value problems. The 1931 edition of Courant and Hilbert's famous book [11] refers to a paper of Bieberbach in Göttingen Nachrichten, 1912 dealing with the elliptic boundary value problem $\Delta u = e^u$ in Ω , $u|_{\partial\Omega} = f$, in a planar region Ω . A solution is found by iterating a monotone map in a function space. Courant and Hilbert extended this method to a broad class of such problems. Out of this technique grew the method of "upper and lower solutions"

(or “supersolutions and subsolutions”) for solving, both theoretically and numerically, second order elliptic PDEs (see Amann [4], Keller and Cohen [39], Keller [40, 41], Sattinger [69]). Krasnoselskii & Zabreiko [45] trace the use of positivity in functional analysis— closely related to monotone dynamics— to a 1924 paper by Uryson [91] on concave operators. The systematic use of positivity in PDEs was pioneered Krasnoselskii & Ladyshenskaya [44] and Krasnoselskii [42] .

Amann [5] showed how a sequence $\{u_n\}$ of approximate solutions to an elliptic problem can be viewed as the trajectory $\{T^n u_0\}$ of u_0 under a certain monotone map T in a suitable function space incorporating the boundary conditions, with fixed points of T being solutions of the elliptic equation. The dynamics of T can therefore be used to investigate the equation. Thus when T is globally asymptotically stable, there is a unique solution; while if T has two asymptotically stable fixed points, in many cases degree theory yields a third fixed point. As Amann [6] emphasized, a few key properties of T — continuity, monotonicity and some form of compactness— allow the theory to be efficiently formulated in terms of monotone maps in ordered Banach spaces.

Many questions in differential equations are framed in terms of eigenvectors of linear and nonlinear operators on Banach spaces. The usefulness of operators that are positive in some sense stems from the theorem of Perron [63] and Frobenius [23], now almost a century old, asserting that for a linear operator on \mathbb{R}^n represented by a matrix with positive entries, the spectral radius is a simple eigenvalue having a positive eigenvector, and all other eigenvalues have smaller absolute value and only nonpositive eigenvectors. In 1912 Jentzsch [37] proved the existence of a positive eigenfunction with a positive eigenvalue for a homogeneous Fredholm integral equation with a continuous positive kernel.

In 1935 the topologists Alexandroff and Hopf [2] reproved the Perron-Frobenius theorem by applying Brouwer’s fixed point theorem to the action of a positive $n \times n$ matrix on the space of lines through the origin in \mathbb{R}_+^n . This was perhaps the first explicit use of the dynamics of operators on a cone to solve an eigenvalue problem. In 1940 Rutman [68] continued in this vein by reproving Jentsch’s theorem by means of Schauder’s fixed point theorem, also obtaining an infinite-dimensional analog of Perron-Frobenius, known today as the Krein-Rutman theorem [48, 85]. In 1957 G. Birkhoff [7] initiated the dynamical use of Hilbert’s projective metric for such questions.

Monotone linear operators such as the Koopman operator and Frobenius-Perron operators arise naturally in the study of ergodicity and mixing properties of measure-preserving transformations. See e.g. Lasota and Mackey [51].

The dynamics of cone-preserving operators continues to play an important role in functional analysis; for a survey, see Nussbaum [60, 61]. One outgrowth of this work has been a focus on purely dynamical questions about such operators; some of these results are presented below. Polyhedral cones in Euclidean spaces have lead to interesting quantitative results, including *a priori* bounds on the number of periodic orbits. For recent work see Lemmens *et al.* [52], Nussbaum [62], Krause and Nussbaum [47], and references therein.

Monotone maps frequently arise as mathematical models in population biology. The matrix demographic model of Leslie [53] are well-known. Nonlinear generalizations of these have also been introduced. See Cushing [14] for a review. Unfortu-

nately, these rarely give rise to monotone maps.

The Leslie-Gower model [54] of two-species competition, given by

$$\begin{aligned} x_{t+1} &= b_1 \frac{1}{1 + c_{11}x_t + c_{12}y_t} x_t \\ y_{t+1} &= b_2 \frac{1}{1 + c_{21}x_t + c_{22}y_t} y_t, \end{aligned} \tag{3}$$

has been analyzed by Cushing et. al. [15] who show that it has the same four dynamical outcomes as the classical autonomous Lotka-Volterra ordinary differential equations model of competition. Expressing (3) as $(x_{t+1}, y_{t+1}) = T(x_t, y_t)$, it can be shown that T is monotone with respect to the order induced by the fourth-quadrant cone.

Planar monotone maps have received considerable attention in the literature because the results for planar maps are considerably stronger than for higher dimensional monotone maps. See Smith [77]. Consequently, we will not devote special attention to planar maps in this survey.

More elaborate two species competition models involving stage structured populations have been studied by Cushing et. al. [15]. The resulting map is also monotone on \mathbb{R}_+^3 under certain restrictions.

Spatially explicit integro-difference equation models considered by Weinberger [95], Liu [56], Weinberger et. al. [96], Allen et. al. [1], and Hart and Gardner [27] give rise to monotone maps. An example is the model of two species competition and dispersal given by:

$$\begin{aligned} p_{t+1}(x) &= \int_{\mathbb{R}^n} b_1 \frac{p_t(x-y)}{1 + c_{11}p_t(x-y) + c_{12}q_t(x-y)} d\mu_1(y) \\ q_{t+1}(x) &= \int_{\mathbb{R}^n} b_2 \frac{q_t(x-y)}{1 + c_{21}p_t(x-y) + c_{22}q_t(x-y)} d\mu_2(y), \end{aligned} \tag{4}$$

where μ_i , probability measures, are referred to as dispersal kernels. This system gives rise to a monotone map T on the positive cone in the product space $C(\mathbb{R}^n) \times C(\mathbb{R}^n)$ relative to the order induced by the cone $C_+(\mathbb{R}^n) \times (-C_+(\mathbb{R}^n))$, where $C(\mathbb{R}^n)$ is the Frechet space of continuous functions with the topology of uniform convergence on compact sets. The variable x could also be discrete, taking values in the integer lattice, if the measures are concentrated there. The map T seems to lack compactness properties required by many results of the theory of monotone maps reviewed here. Attention focuses on the existence of traveling waves of invasion joining two equilibria.

For monotone maps as models for the spread of a gene or an epidemic through a population see Thieme [90], Selgrade and Ziehe [70], and the references therein.

2 Definitions and Basic Results

An *ordered Banach space* is a Banach space Y with cone Y_+ , a closed subset of Y with the properties: $\mathbb{R}_+ \cdot Y_+ \subset Y_+$, $Y_+ + Y_+ \subset Y_+$, $Y_+ \cap (-Y_+) = \{0\}$. We always assume $Y_+ \neq \{0\}$. Y_+ is viewed as the set of positive elements in Y :

$y \in Y_+ \iff y \geq 0$. As usual, if $x, y \in Y$, we write $x \leq y$ ($x < y$) if $y - x \in Y_+$ ($y - x \in Y_+ \setminus \{0\}$). When $\text{Int}Y_+$ is nonempty we call Y a *strongly ordered* Banach space. In this case $x \ll y \iff y - x \in \text{Int}Y_+$. Occasionally it is useful to make use of the dual cone $(Y_+)^* = \{y^* \in Y^* : y^*(Y_+) \subset [0, \infty)\}$, which may not be a cone since $(Y_+)^* \cap -(Y_+)^* = \{0\}$ may fail. Vector y belongs to Y_+ if and only if $y^*(y) \geq 0$ for all $y^* \in (Y_+)^*$; if Y is strongly ordered, then $y \in \text{Int}Y_+$ implies $y^*(y) > 0$ for all nontrivial $y^* \in (Y_+)^*$ and $y \in \partial Y_+$ implies that $y^*(y) = 0$ for some nontrivial $y^* \in (Y_+)^*$. See [19].

If $x < y$, we write $[x, y] = \{z \in Y : x \leq z \leq y\}$ for the order interval with endpoints x and y ; we write $[[x, y]] = \{z \in Y : x \ll z \ll y\}$ for the open order interval in strongly ordered spaces. Subset X of Y is *order bounded* if it is contained in some order interval. A subset $X \subset Y$ is *p-convex* if it contains the line segment joining any two of its points x and y with $x < y$; it is *order convex* if it contains $[x, y]$ for any two of its points x and y with $x < y$. A subset A of Y is *unordered* if it does not contain points x, y with $x < y$. $A \leq B$ for subsets A, B of Y means $a \leq b$ for every pair $a \in A, b \in B$.

A continuous map $T : X \rightarrow X$ on the subset $X \subset Y$ is

1. *monotone* if $x \leq y \Rightarrow Tx \leq Ty$
2. *strictly monotone* if $x < y \Rightarrow Tx < Ty$
3. *strongly monotone* if $x < y \Rightarrow Tx \ll Ty$
4. *eventually strongly monotone* if $x < y$, there exists $n_0 \geq 1$ such that $T^n x \ll T^n y, n \geq n_0$.
5. *strongly order-preserving (SOP)* if T is monotone, and when $x < y$ there exist respective neighborhoods U, V of x, y and $n_0 \geq 1$ such that $n \geq n_0 \Rightarrow T^n U \leq T^n V$.¹

Obviously, strong monotonicity is restricted to strongly ordered Banach spaces but SOP is not. It is easy to see that eventual strong monotonicity implies the strong order preserving property. Slightly different terminology is used if the map T is linear. A linear operator $T \in L(Y)$ is called *positive* if $T(Y_+) \subset Y_+$ and *strongly positive* if $T(Y_+ \setminus \{0\}) \subset \text{Int}Y_+$.

A fundamental result for compact, strongly positive linear operators is the Krein-Rutman Theorem.

Theorem 2.1 (*Krein-Rutman*) *Let A be a compact, strongly positive linear operator on the Banach space Y and set $r = \rho(A)$. Then Y decomposes into a direct sum of two closed invariant subspaces Y_1 and Y_2 such that $Y_1 = N(A - rI)$ is spanned by*

¹Our use of “strongly order-preserving” conflicts with Dancer & Hess [17], who use these words to mean what we have defined as “strongly monotone”. Our usage is consistent with that of several authors. Takáč [80, 81] uses “strongly increasing” for our SOP.

$z \gg 0$ and $Y_2 \cap Y_+ = \{0\}$. Moreover, the spectrum of $A|Y_2$ is contained in the closed ball of radius $\nu < r$ in the complex plane.

See Krein and Rutman [48], Takáč [85] or Zeidler [99] for proofs.

The orbit of x is $O(x) := \{T^n x\}_{n \geq 0}$, and the omega limit set of x is $\omega(x) := \bigcap_{k \geq 0} \overline{O(T^k x)}$. If $O(x)$ has compact closure, $\omega(x)$ is nonempty, compact, invariant (that is, $T\omega(x) = \omega(x)$) and *invariantly connected*. The latter means that $\omega(x)$ is not the disjoint union of two closed invariant sets [50].

If $T(x) = x$ then x is a *fixed point* or *equilibrium*. E denotes the set of fixed points. More generally, if $T^k x = x$ for some $k \geq 1$ we call x *periodic*, or k -periodic. The minimal such k is called the period of x (and $O(x)$).

The following result is useful for proving smooth maps monotone or strongly monotone:

Lemma 2.2 *Let $X \subset Y$ be a p -convex open set, $f : X \rightarrow Y$ a C^1 map with $f'(x)$ positive for all $x \in X$. Then f is monotone. If, in addition, for each $x, y \in X$ with $x < y$, there exists z on the line segment joining x and y such that $f'(z)$ is strongly positive, then f is strongly monotone.*

Proof By p -convexity, if $x, y \in X$ and $x < y$ then the line segment joining them belongs to X so

$$f(y) - f(x) = \int_0^1 f'(ty + (1-t)x)(y-x)dt \geq 0$$

because $f'(z)$ is positive for $z \in X$. If it is also strongly positive at some $ty + (1-t)x$, then for nontrivial $y^* \in (Y_+)^*$ we have $y^*(f'(ty + (1-t)x)(y-x)) > 0$ so it follows that the right side is positive. As $y^* \in (Y_+)^* \setminus \{0\}$ is arbitrary, it follows that $f(y) - f(x) \gg 0$. ■

In the special case that $Y = \mathbb{R}^n$ and Y_+ is an orthant $K_m := \{x \in \mathbb{R}^n : (-1)^{m_i} x_i \geq 0\}$, where $m = (m_1, m_2, \dots, m_n)$, $m_i \in \{0, 1\}$, there is a simple test for a map $f : X \rightarrow X$, where $X \subset \mathbb{R}^n$ is p -convex, to be monotone with respect to K_m for some m , in terms of the Jacobian $Df(x)$. The following must hold:

- (a) Sign stability: $\forall i, j$, $\frac{\partial f_i}{\partial x_j}(x)$ does not change sign in X .
- (b) Sign symmetry: $\forall i, j, \forall x, y \in X$ $\frac{\partial f_i}{\partial x_j}(x) \frac{\partial f_j}{\partial x_i}(y) \geq 0$.
- (c) Sign consistency: the signed incidence graph associated with the Jacobian matrix $Df(x)$, the graph with undirected edge joining vertices i and j if $\frac{\partial f_i}{\partial x_j}(x) + \frac{\partial f_j}{\partial x_i}(y) \neq 0$ for some x, y , the edge being given the sign of this sum, has the property that every loop has an even number of negative signs (negative feedbacks).

Note in particular that by taking $i = j$ in (c) with loop $i \rightarrow i$, the diagonal entries of $Df(x)$ must be nonnegative: $\frac{\partial f_i}{\partial x_i}(x) \geq 0$. See [75] for further elaboration in case of differential equations.

We begin with the basic results of the theory. In order to state the following result succinctly, let $N_0 := \mathbb{N} \cup \{0\}$. For $a, b \in N_0$, $a < b$ we set $[a, b] = \{j \in N_0 : a \leq j \leq b\}$ (there will be no confusion with real intervals). Let $J \subset N_0$ be an interval and $f : J \rightarrow X$ be a map. A subinterval $[a, b] \subset J$, $a < b$ is *rising* if $f(a) < f(b)$, and *falling* if $f(b) < f(a)$.

It is well known that a monotone map $T : \mathbb{R} \rightarrow \mathbb{R}$ (or on any totally ordered set) has the property that every trajectory $n \mapsto T^n x$ is either increasing or decreasing. In particular, a trajectory cannot have both a rising and a falling interval. The following result says this holds for all monotone maps; note that topology plays no role.

Theorem 2.3 (NO ORBIT RISES AND FALLS) *A trajectory of a monotone map $T : X \rightarrow X$ cannot have both a rising interval and a falling interval.*

Proof: Fix a trajectory $f : N_0 \rightarrow X$, $f(n) := T^n x$. Call an interval $[m, n]$ *weakly falling* if $f(n) \leq f(m)$. Right translates of weakly falling intervals are weakly falling by monotonicity.

We proceed by contradiction. Let $[a, a+j]$ be a falling interval of minimal length and $[b, b+k]$ a rising interval of minimal length. We assume $a \leq b$, the case $b \leq a$ being similar.

The interval $[b, b+j]$, a right translate of $[a, a+j]$, is weakly falling. Therefore $j \neq k$, since otherwise $[b, b+k]$ would be both rising and weakly falling. If $j < k$, we get the contradiction that $[b+j, b+k]$ is a rising interval of length $k-j < k$, because $f(b+j) \leq f(b) < f(b+k)$. But if $j > k$ we get the contradiction that $[b+k, b+j]$ is a falling interval of length $j-k < j$, because $f(b+k) > f(b) \geq f(b+j)$. ■

Theorem 2.3 appears first in Hirsch [31] for solutions of odes, its proof credited to L. Ito. A revised proof appears in [75] and one valid for both semiflows and mappings is presented in [34].

Proposition 2.4 (NONORDERING OF PERIODIC ORBITS) *A periodic orbit of a monotone map is unordered.*

Proof: If not, there exists x in the orbit such that $T^k(x) > x = T^{k+l}x$ for some $k, l > 0$, contradicting Theorem 2.3. ■

The next result uses the simple fact that a monotone sequence in a compact set converges.

Lemma 2.5 (MONOTONE CONVERGENCE CRITERION) *Assume T is monotone and $O(z)$ has compact closure. If $m \geq 1$ is such that $T^m z < z$ or $T^m z > z$ then $\omega(z)$ is an m -periodic orbit.*

Proof: Consider first the case $m = 1$. Compactness of $\overline{O(z)}$ implies the decreasing sequence $\{T^k z\}$ converges to a point $p = \omega(z)$. Now suppose $m > 1$. Applying the case just proved to the map T^m , we conclude that $\{T^{km} z\}$ converges to a point $p = T^m(p)$. It follows that $\omega(z) = \{p, Tp, T^2p, \dots, T^{m-1}p\}$. ■

An immediate corollary is that a precompact orbit containing two ordered points is asymptotic to (or equal to) a periodic orbit:

Corollary 2.6 *Assume T is monotone and $O(z)$ has compact closure. If $T^m z \leq T^n z$ for some m, n , $m \neq n$, then $\omega(z)$ is a periodic orbit.*

The next result is fundamental to the theory of monotone maps:

Theorem 2.7 (NONORDERING PRINCIPLE) *Let $\omega(z)$ be an omega limit set for a monotone map T .*

(i) *No points of $\omega(z)$ are related by \ll .*

(ii) *If $\omega(z)$ is a periodic orbit or T is SOP, no points of $\omega(z)$ are related by $<$.*

Proof: (i) Follows from Theorem 2.3; first part of (ii) is a consequence of Proposition 2.4. If T is SOP, $a, b \in \omega(z)$, $a < b$, then there are neighborhoods U, V of a, b , respectively, and n_0 such that $T^{n_0}(U) \leq T^{n_0}(V)$. As $T^n(z)$ must repeatedly meet U and V , it follows that either it must have both a rising and falling interval or it is eventually periodic. Either alternative gives a contradiction ■

Generic convergence fails for strongly monotone maps. We now point out a significant difference between strongly monotone maps and semiflows: *The Limit Set Dichotomy fails for strongly monotone maps.* Recall that for an SOP semiflow with compact orbit closures, the Limit Set Dichotomy (LSD) states:

If $a < b$, either $\omega(a) < \omega(b)$ or $\omega(a) = \omega(b) \subset E$.

This result was first proved in [33] for strongly monotone semiflows, later stated by Matano [58] and proved in Smith and Thieme for SOP semiflows [78]. See [34] for a simple proof. Most all of the important results in the theory of monotone semiflows follow from this deceptively simple result, perhaps with additional assumptions. In particular, it implies that the generic orbit converges to the set of equilibria and that there cannot be an attracting periodic orbit. See [33, 78, 75, 34].

Takac [83], Theorem 3.10, gives conditions on strongly monotone maps under which $a < b$ implies that either $\omega(a) \cap \omega(b) = \emptyset$ or $\omega(a) = \omega(b)$. He also gives a counterexample showing that $\omega(a) \cap \omega(b) = \emptyset$ does not imply $\omega(a) < \omega(b)$, nor does $\omega(a) = \omega(b)$ imply that these limit sets consist of fixed points (they are period-two orbits in his example). However, the mapping in his example is defined on a disconnected space.

LSD must fail for any map T in a Banach space, having an asymptotically stable periodic point p of period > 1 : take a point $q > p$ so near to p that $O(p) = \omega(p) = \omega(q)$. Clearly $\omega(p)$, being a nontrivial periodic orbit, contains no fixed points. Thus the second assertion of LSD fails in this case.

Dancer and Hess [17] gave a simple example in \mathbb{R}^k for prime k of a strongly monotone map with an asymptotically stable periodic point of period k which we describe below. Therefore the second alternative of LSD can be no stronger than that $\omega(a) = \omega(b)$ is a periodic orbit.

LSD fails for strictly monotone maps in \mathbb{R}^2 . To see this, define $T_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T_0(x, y) := (f(y), f(x)), \quad f(x) = 2 \arctan(x).$$

Let $a > 0$ be the unique positive fixed point of f and note that $0 < f'(a) < 1$. Then the set of fixed points of T_0 is $E = \{(-a, -a), (0, 0), (a, a)\}$ since f has no points of period 2. The fixed points of T_0^2 are the nine points obtained by taking all pairings of $-a, 0, a$. An easy calculation shows that $\{(-a, a), (a, -a)\}$ is an asymptotically stable period-two orbit of T_0 because the Jacobian matrix of T_0^2 is $f'(a)^2$ times the identity matrix. Thus, LSD fails. However, T_0 is strictly monotone but not strongly monotone. Now consider the perturbations

$$T_\epsilon(x, y) := T_0(x, y) + \epsilon(x, y).$$

It is easy to see that T_ϵ is strongly monotone for $\epsilon > 0$. By the implicit function theorem, for small $\epsilon > 0$, T_ϵ has an asymptotically stable period-two orbit $O(p_\epsilon)$ with p_ϵ near $(-a, a)$. As noted in [17], this example can be generalized to \mathbb{R}^k for prime k .

Takáč [84] shows that linearly stable periodic points can arise for the period map associated with monotone systems of ordinary and partial differential equations. Other counterexamples for low dimensional monotone maps can be found in Smith [77, 76]. For example, the dynamics of any one-dimensional recursion $x_{n+1} = h(x_n)$, where $h(x) = f(x) - g(x)$, f, g strictly increasing functions on \mathbb{R} , can be embedded on the negative diagonal $y = -x$ for the strongly monotone map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (f(x) - g(-y), g(x) - f(-y))$. Consequently, there are planar strongly monotone maps having arbitrarily complicated one-dimensional dynamics on $y = -x$. One can show that for any positive integer q , there is such a planar map T which has an asymptotically stable periodic orbit of period q [76].

3 Attractors contain attracting periodic orbits

As we have shown, asymptotically stable periodic orbits that are not singletons can exist for monotone, even strongly monotone maps. Later we will show that the generic orbit of a smooth, dissipative, strongly monotone map converges to a periodic orbit. Here, we show that every attractor contains a periodic orbit that attracts an open set.

A set K attracts a point y if $\overline{O(y)}$ is compact and $\omega(y) \subset K$; K attracts a set U if it attracts each point of U . Recall that a point p is *wandering* if there exists a neighborhood U of p and a positive integer n_0 such that $T^n(U) \cap U = \emptyset$ for $n > n_0$. The *nonwandering set* Ω , consisting of all points q that are not wandering, contains all limit sets. The following result is due to Hirsch, Theorem 4.1 [32].

Theorem 3.1 *Let X be an open subset of the strongly ordered Banach space Y and $T : X \rightarrow X$ be monotone with compact orbit closures. If K is a compact invariant set that attracts some neighborhood of itself, then K contains a periodic orbit that attracts an open set.*

Proof: $K \cap \Omega$ is nonempty and compact so we may choose a maximal element p of it. Suppose K attracts the open neighborhood U of K and fix $y \gg p$, $y \in U$. Since p is nonwandering there exists a convergent sequence $x_i \rightarrow p$ and a sequence $n_i \rightarrow \infty$ such that $T^{n_i}x_i \rightarrow p$. For all large i , $x_i \leq y$. Passing to a subsequence, we assume that $T^{n_i}y \rightarrow q$. By monotonicity and $x_i \leq y$ for large i , we have $q \geq p$. But $q \in K \cap \Omega$ and the maximality of p requires $q = p$. Since $p \ll y$ and $T^{n_i}y \rightarrow p$ it follows that $T^m y \ll y$ for some m . Lemma 2.5 implies that $\omega(y)$ is an m -periodic orbit containing p . As this holds for every $y \gg p$, the result follows. \blacksquare

If T is strongly monotone, Hirsch, Theorem 6.3 [32] proves the existence of a periodic orbit in K that is stable relative to the topology generated by open order intervals $[[a, b]]$. If, in addition, K is a uniform attractor of some neighborhood of itself, then it must contain a stable periodic orbit, Hirsch and Smith [34]. A result of Jiang and Yu, Theorem 2, [38] then shows that if T is analytic, order compact with strongly positive derivative, K must contain an asymptotically stable periodic orbit.

The following result constrains the size of the domain of attraction of a nontrivial periodic orbit.

Proposition 3.2 *Let A be an invariant set for an SOP map T that attracts an upper bound x . Then $a := \sup A$ exists, belongs to A , and $\omega(x) = a$ so $Ta = a$. An analogous statement holds if A attracts a lower bound. A non-trivial periodic orbit cannot attract an upper or lower bound.*

Proof: $A \leq x$ implies $A \leq \omega(x)$ by invariance of A . As $\omega(x) \subset A$, we have $\omega(x) \leq \omega(x)$ implying that $\omega(x)$ is a singleton by the Nonordering Principle and SOP. So $\omega(x) = a \in A$ and $A \leq a$ implying that $a = \sup A$.

If a nontrivial periodic orbit O attracts an upper bound, then $\sup O$ exists and is a fixed point, a contradiction. \blacksquare

4 Dynamics of Smooth strongly monotone maps

The generic orbit of a smooth strongly order preserving semiflow converges to fixed point but such a result fails to hold for discrete semigroups, i.e., for strongly order preserving mappings. Indeed, such mappings can have attracting periodic orbits of period exceeding one as we have just seen. However, Tereščák [88], improving earlier joint work with Poláčik [66, 67], and [30], has obtained the strongest result possible for strongly monotone, smooth, dissipative mappings. Recall that map T is *point dissipative* (see Hale [24]) provided there is a bounded set B with the property that for every $x \in X$, there is a positive integer $n_0 = n_0(x)$ such that $T^n x \in B$ for all $n \geq n_0$.

Theorem 4.1 (Tereščák, 1994) *Let $T : Y \rightarrow Y$ be a completely continuous, C^1 , point dissipative map whose derivative is strongly positive at every point of the ordered Banach space Y having cone Y_+ with nonempty interior. Then there is a*

positive integer m and an open dense set $U \subset Y$ such that the omega limit set of every point of U is a periodic orbit with period at most m .

We note that the hypothesis that $T'(x)$ is strongly positive implies that T is strongly monotone by Lemma 2.2. It is unfortunate that Tereščák's Theorem has not yet been published.

Smoothness together with compactness allows one to settle questions of stability of fixed points and periodic points by examining the spectrum of the linearization of the mapping. Let $T : X \rightarrow X$ where X is an open subset of the ordered Banach space Y with cone Y_+ having nonempty interior in Y . Assume that T is a completely continuous, C^1 mapping with a strongly positive derivative at each point. Then T is strongly monotone by Lemma 2.2 and $T'(x)$ is a Krein-Rutman operator so the Krein-Rutman Theorem 2.1 holds for $T'(p)$, $p \in E$. Let ρ be the spectral radius of $T'(p)$, which the reader will recall is a simple eigenvalue which dominates all others in modulus and for which the generalized eigenspace is spanned by an eigenvector $v \gg 0$. Let V_1 be the span of v in Y . There is a complementing closed subspace V_2 such that $Y = V_1 \oplus V_2$ satisfying $T'(p)V_2 \subset V_2$ and $V_2 \cap Y_+ = \{0\}$. Let P denote the projection of Y onto V_2 along v . Finally, let τ denote the spectral radius of $T'(p)|_{V_2} : V_2 \rightarrow V_2$, which obviously satisfies $\tau < \rho$. Mierczyński [59] exploits this structure of the linearized mapping to obtain very detailed behavior of the orbits of points near p . In order to describe his results, define $K := \{x \in X : T^n x \rightarrow p\}$ to be the basin of attraction of p . Let $M_- := \{x \in X : T^{n+1}x \ll T^n x, n \geq n_0, \text{ some } n_0\}$ be the set of eventually decreasing orbits, $M_+ := \{x \in X : T^n x \ll T^{n+1}x, n \geq n_0, \text{ some } n_0\}$ be the set of eventually increasing orbits, and $M := M_- \cup M_+$ be the set of eventually monotone (in the strong sense) orbits.

The following result is standard but nonetheless important.

Theorem 4.2 (PRINCIPLE OF LINEARIZED STABILITY) *If $\rho < 1$, there is a neighborhood U of p such that $T(U) \subset U$ and constants $c > 0$, $\kappa \in (\rho, 1)$ such that for each $x \in U$ and all n*

$$\|T^n x - p\| \leq c\kappa^n \|x - p\|.$$

In the more delicate case that $\rho \leq 1$, Mierczyński [59] obtains a smooth hypersurface C , which is an analog for T of the codimension-one linear subspace V_2 invariant under the linearized mapping $T'(p)$:

Theorem 4.3 *If $\rho \leq 1$ there exists a codimension-one embedded invariant manifold $C \subset X$ of class C^1 having the following properties:*

- (i) $C = \{p + Pw + R(w)v : w \in O\}$ where $R : O \rightarrow \mathbb{R}$ is a C^1 map defined on the relatively open subset O of V_2 containing 0, satisfying $R(0) = R'(0) = 0$. In particular, C is tangent to V_2 at p .
- (ii) C is unordered.
- (iii) $C = \{x \in X : \|T^n x - p\|/\kappa^n \rightarrow 0\} = \{x \in X : \|T^n x - p\|/\kappa^n \text{ is bounded}\}$, for any κ , $\tau < \kappa < \rho$. In particular, $C \subset K$.
- (iv) $K \setminus C = \{x \in K : \|T^n x - p\|/\kappa^n \rightarrow \infty\} = \{x \in K : \|T^n x - p\|/\kappa^n \text{ is unbounded}\}$, for any κ , $\tau < \kappa < \rho$.

(v) $K \setminus C = K \cap M$.

Conclusion (v) implies most orbits converging to p do so monotonically, but more can be said. Indeed, $K \cap M_+ = \{x \in K : (T^n x - p) / \|T^n x - p\| \rightarrow -v\}$ and a similar result for $K \cap M_-$ with v replacing $-v$ holds. The manifold C is a local version of the unordered invariant hypersurfaces obtained by Takáč in [81].

Corresponding to the space V_1 spanned by $v \gg 0$ for $T'(p)$, a locally forward invariant, one dimensional complement to the codimension one manifold C is given in the following result.

Theorem 4.4 *There is $\epsilon > 0$ and a one-dimensional locally forward invariant C^1 manifold $W \subset B(p; \epsilon)$, tangent to v at p . If $\rho > 1$, then W is locally unique, and for each $x \in W$ there is a sequence $\{x_{-n}\} \subset W$ with $Tx_{-n} = x_{-n+1}$, $x_0 = x$, and $\kappa^n \|x_{-n} - p\| \rightarrow 0$ for any κ , $1 < \kappa < \rho$.*

Here $B(p; \epsilon)$ is the open ϵ -ball centered at p . Local forward invariance of W means that $x \in W$ and $Tx \in B(x; \epsilon)$ implies $Tx \in W$. Related results are obtained by Smith [72]. In summary, the above results assert that the dynamical behavior of the nonlinear map T behaves near p like that of its linearization $T'(p)$. Obviously, the above results can be applied at a periodic point p of period k by considering the map T^k which has all the required properties.

Mierczyński [59] uses the results above to classify the convergent orbits of T . Similar results are obtained by Takáč in [82].

It is instructive to consider the sort of stable bifurcations that can occur from a linearly stable fixed point, or a linearly stable periodic point, for a one parameter family of mappings satisfying the hypotheses of the previous results, as the parameter passes through a critical value at which $\rho = 1$. The fact that there is a simple positive dominant eigenvalue of $(T^k)'(p)$ ensures that period-doubling bifurcations from a stable fixed point or from a stable periodic point, as a consequence of a real eigenvalue passing through -1 , cannot occur. In a similar way, a Neimark-Sacker [49] bifurcation to an invariant closed curve cannot occur from a stable fixed or periodic point. These sorts of bifurcations can occur from unstable fixed or periodic points but then they will “be born unstable”.

5 The Order Interval Trichotomy

In this subsection we assume that X is a subset of an ordered Banach space Y with positive cone Y_+ , with the induced order and topology. Much of the early work on monotone maps on ordered Banach spaces focused on the existence of fixed points for self maps of order intervals $[a, b]$ such that $a, b \in E$; see especially Amann [6]. The following result of Dancer and Hess [17], quoted without proof, is crucial for analyzing such maps.

Let u, v be fixed points of T . A doubly-infinite sequence $\{x_n\}_{n \in \mathbb{Z}}$ (\mathbb{Z} is the set of all integers) in Y is called an *entire orbit from u to v* if

$$x_{n+1} = T(x_n), \quad \lim_{n \rightarrow -\infty} x_n = u, \quad \lim_{n \rightarrow \infty} x_n = v$$

If $x_n \leq x_{n+1}$ (respectively, $x_n < x_{n+1}$), the entire orbit is *increasing* (respectively, *strictly increasing*). If $x_n \geq x_{n+1}$ (respectively, $x_n > x_{n+1}$), the entire orbit is *decreasing* (respectively, *strictly decreasing*). If the entire orbit $\{x_n\}$ is increasing but not strictly increasing, then $x_n = v$ for all sufficiently large n ; and similarly for decreasing.

Consider the following hypothesis:

(G) $X = [a, b]$ where $a, b \in Y$, $a < b$. The map $T : X \rightarrow X$ is monotone and $T(X)$ has compact closure in X .

Theorem 5.1 (THE ORDER INTERVAL TRICHOTOMY) *Under hypothesis (G), at least one of the following holds:*

- (a) *there is a fixed point c such that $a < c < b$*
- (b) *there exists an entire orbit from a to b that is increasing, and strictly increasing if T is strictly monotone*
- (c) *there exists an entire orbit from b to a that is decreasing, and strictly decreasing if T is strictly monotone*

An extension of Theorem 5.1 to allow additional fixed points on the boundary of $[a, b]$ is carried out in Hsu *et al.* [36]. Wu *et al.* [97] weaken the compactness condition. See Hsu *et al.* [36], Smith [77], and Smith and Thieme [79] for applications to generalized two-species competition dynamics. For related results see Hess [28], Matano [57], Poláčik [64], Smith [72, 75].

A fixed point q of T is *stable* if every neighborhood of q contains a positively invariant neighborhood of q . An immediate corollary of the Order Interval Trichotomy is:

Corollary 5.2 *Assume hypothesis (G), and let a and b be stable fixed points. Then there is a third fixed point in $[a, b]$.*

Corollary 5.4 establishes a third fixed point under different assumptions.

In general, more than one of the alternatives (a), (b), (c) may hold (see [36]). The following complement to the Order Interval Trichotomy gives conditions for exactly one to hold; (iii) is taken from Proposition 2.2 of [36].

Consider the following three conditions:

- (a') *there is a fixed point c such that $a < c < b$*
- (b') *there exists an entire orbit from a to b .*
- (c') *there exists an entire orbit from b to a*

Proposition 5.3 *Assume hypothesis (G).*

- (i) *If T is strongly order-preserving, exactly one of (a'), (b'), (c') can hold. More precisely: Assume $a < y < b$ and y has compact orbit closure. Then $\omega(y) = \{b\}$ if there is an entire orbit from a to b , while $\omega(y) = \{a\}$ if there is an entire orbit from b to a .*

(ii) If $a \ll b$, at most one of (b'), (c') can hold.

(iii) Suppose $a \ll b$, and $E \cap [a, b] \setminus \{a, b\} \neq \emptyset$ implies $E \cap [[a, b]] \setminus \{a, b\} \neq \emptyset$. Then at most one of (a'), (b'), (c') can hold.

Proof: For (i), consider an entire orbit $\{x_n\}$ from a to b . There is a neighborhood U of a such that $T^k U \leq T^k y$ for sufficiently large k . Choose $x_j \in U$. Then $T^k x_j \leq T^k y \leq b$ for all large k . As $\lim_{k \rightarrow \infty} T^k x_j = b$ and the order relation is closed, b is the limit of every convergent subsequence of $\{T^k y\}$. The case of an entire orbit from b to a is similar.

In (ii), choose neighborhoods U, V of a, b respectively such that $U \ll V$. Fix j so that $x_j \in U$. If $y \in V$ then an argument similar to the proof of (i) shows that $\omega(y) = \{b\}$. Hence there cannot be an entire orbit from b to a , since it would contain a point of V .

Assume the hypothesis of (iii), and note that (ii) makes (b') and (c') incompatible. If (a'), there is a fixed point $c \in [[a, b]]$, and arguments similar to the proof of (ii) show that neither (b') nor (c') holds. ■

Corollary 5.4 *In addition to hypothesis (G), assume T is strongly order preserving. If some trajectory does not converge, there is a third fixed point.*

Proof Follows from the Order Interval Trichotomy Theorem 5.1 and Proposition 5.3(i). ■

A number of authors have considered the question of whether *a priori* knowledge that every fixed point is stable implies the convergence of every trajectory. See Alikakos *et al.* [3], Dancer and Hess [17], Matano [57] and Takáč [81] for such results. The following theorem, adapted from [17], is proved in Hirsch and Smith [34].

A set $A \subset X$ is a *uniform global attractor* for the map $T : X \rightarrow X$ if $T(A) = A$ and $\text{dist}(T^n x, A) \rightarrow 0$ uniformly in $x \in X$.

Theorem 5.5 *Let $a, b \in Y$ with $a < b$. Assume $T : [a, b] \rightarrow [a, b]$ is strongly order preserving with precompact image, and every fixed point is stable. Then E is a totally ordered arc J that is a uniform global attractor, and every trajectory converges.*

If the map T in Theorem 5.5 is C^1 and strongly monotone, then E is a smooth totally ordered arc by a result of Takáč [83].

Existence of fixed points

Dancer [18] obtained remarkable results concerning the dynamics of monotone maps with some compactness properties: Limit sets can always be bracketed between two fixed points, and with additional hypotheses these fixed points can be chosen to be stable. These results do not require smoothness nor strong monotonicity. The next two theorems are adapted from [18].

A map $T : Y \rightarrow Y$ is *order compact* if it takes each order interval, and hence each order bounded set, into a precompact set.

Theorem 5.6 *Let $X \subset Y$ and $T : X \rightarrow X$ be monotone such that every orbit has compact closure in X . If either*

(a) *X is order convex, T is order compact, and every limit set order bounded, or*

(b) *every limit set has an infimum and supremum in X*

then for all $z \in X$ there are fixed points f, g such that $f \leq \omega(z) \leq g$.

Proof: (a): There exists $u \in X$ such that $u \geq \omega(z)$ because omega limit sets order bounded. Since $T(\omega(z)) = \omega(z)$, it follows that $\omega(z) \leq T^i u$ for all i , hence $\omega(z) \leq \omega(u)$. Similarly, there exists $s \in X$ such that $\omega(u) \leq \omega(s)$. The set $F := \{x \in Y : \omega(z) \leq x \leq \omega(s)\}$ is the intersection of closed order intervals, hence closed and convex, non-empty because it contains $\omega(u)$, and obviously order bounded. Moreover $F \subset X$ because X is order convex. Therefore $T(F)$ is defined and is precompact. Monotonicity of T and invariance of $\omega(z)$ and $\omega(s)$ imply $T(F) \subset F$. It follows from the Schauder fixed point theorem that there is a fixed point $g \in F$, and $g \geq \omega(z)$ as required. The existence of f is proved similarly.

(b): If $a = \inf \omega(z)$, then $a \leq \omega(z)$ and monotonicity and invariance of $\omega(z)$ imply $Ta \leq \omega(z)$. The definition of a implies $Ta \leq a$ and consequently $\omega(a) = \{f\}$ with $f \leq \omega(z)$. The existence of g is proved similarly. ■

The cone Y_+ is *reproducing* if $Y = Y_+ - Y_+$. This holds for many function spaces whose norms do not involve derivatives. If Y_+ has nonempty interior, it is reproducing: any $x \in Y$ can be expressed as $x = \lambda e - \lambda(e - \lambda^{-1}x) \in Y_+ - Y_+$, where $e \gg 0$ is arbitrary and $\lambda > 0$ is a sufficiently large real number.

Theorem 5.7 *Let $X \subset Y$ be order convex. Assume $T : X \rightarrow X$ is monotone, completely continuous, and order compact. Suppose orbits are bounded and omega limits sets are order bounded.*

(i) *For all $z \in X$ there are fixed points f, g such that $f \leq \omega(z) \leq g$.*

(ii) *Assume Y_+ is reproducing, $X = Y$ or Y_+ , and E is bounded. Then there are fixed points $e_M = \sup E$ and $e_m = \inf E$, and all omega limit sets lie in $[e_m, e_M]$. Moreover, if $x \leq e_m$ then $\omega(x) = \{e_m\}$, while if $x \geq e_M$ then $\omega(x) = \{e_M\}$.*

(iii) *Assume Y_+ is reproducing, $X = Y$ or Y_+ , E is bounded, and T is strongly order preserving. Suppose $z_0 \in Y$ is not convergent. Then there are three fixed points $f < p < g$ such that $f < \omega(z_0) < g$. If T is strongly monotone, f and g can be chosen to be stable.*

Proof: We prove all assertions except for the stability in (iii). Complete continuity implies that every positively invariant bounded set is precompact. Therefore orbit closures are compact and omega limit sets are compact and nonempty, so (i) follows from Theorem 5.6.

To prove (ii), note that E is compact because it is bounded invariant and closed. Compact sets have maximal (minimal) elements; choose a maximal element $e_M \in E$. We must show that $e_M \geq e$ for every $e \in E$. Since the order cone is reproducing, $e_M - e = v - w$ with $v, w \geq 0$. Set $u := e + v + w$. Then $u \in X$, $u \geq e$, and $u \geq e_M$. Monotonicity implies $e_M = T^i e_M \leq T^i u$ for all $i \geq 0$, hence $e_M \leq \omega(u)$. By Theorem 5.6 there exists $g \in E$ such that $\omega(u) \leq g$. Hence $e_M \leq g$, whence $e_M = g$ by maximality. We now have $e_M \leq \omega(u) \leq g = e_M$, so $\omega(u) = \{e_M\}$. Monotonicity implies (as above) $e \leq \omega(u)$, therefore $e \leq e_M$ as required. This proves $e_M = \sup E$, and the dual argument proves $e_m = \inf E$. If $x \leq e_m$ then $\omega(x) \leq e_m$ by monotonicity; but $\omega(x) \geq e_m$ by (i), so $\omega(x) = \{e_m\}$. Similarly for the case $x \geq e_M$.

To prove the first assertion of (iii), note that $e_m < \omega(z) < e_M$ by (i) and the Nonordering Principle 2.7(ii). Monotonicity and order compactness of T imply $[e_m, e_M]$ is positively invariant with precompact image. As T is SOP, there is a third fixed point in $[e_m, e_M]$ by Corollary 5.4. \blacksquare

Remark 5.8 *Under the hypotheses of Theorem 5.7(iii), if E does not contain three ordered fixed points, every orbit converges to a fixed point. Takáč, Theorem 0.2 [83] shows that if there are not three ordered periodic points of T , then every limit set is a periodic orbit.*

6 Sublinearity and the Cone Limit Set Trichotomy

Motivated by the problem of establishing the existence of periodic solutions of quasimonotone, periodic differential equations defined on the positive cone in R^n , Krasnoselskii pioneered the dynamics of sublinear monotone self-mappings of the cone [43]. We will prove Theorem 6.3 below, adapted from the original finite-dimensional version of Krause and Ranft [46].

Let Y denote an ordered Banach space with positive cone Y_+ . Denote the interior (possibly empty) of Y_+ by P . A map $T : Y_+ \rightarrow Y_+$ is *sublinear* (or “subhomogeneous”) if

$$0 < \lambda < 1 \Rightarrow \lambda T(x) \leq T(\lambda x),$$

and *strongly sublinear* if

$$0 < \lambda < 1, \quad x \gg 0 \Rightarrow \lambda T(x) \ll T(\lambda x)$$

Strong sublinearity is the strong concavity assumption of Krasnoselskii [43]. It can be verified by using the following result from that monograph:

Lemma 6.1 *Let $P \neq \emptyset$. $T : P \rightarrow Y$ is strongly sublinear provided T is differentiable and $Tx \gg T'(x)x$ for all $x \gg 0$.*

Proof: Let $F(s) = s^{-1}T(sx)$ for $s > 0$ and some fixed $x \gg 0$. Then $F'(s) = -s^{-2}T(sx) + s^{-1}T'(sx)x \ll 0$ by our hypothesis. So for $0 < \lambda < 1$, we have

$$\phi(Tx - \lambda^{-1}T(\lambda x)) = \phi(F(1)) - \phi(F(\lambda)) < 0$$

for every nontrivial $\phi \in Y_+^*$, the dual cone in Y^* , because $\frac{d}{ds}\phi(F(s)) < 0$. It follows that $Tx - \lambda^{-1}T(\lambda x) \ll 0$. \blacksquare

Corollary 6.2 *Assume Y is strongly ordered. A continuous map $T : Y_+ \rightarrow Y_+$ is sublinear provided T is differentiable in P and $Tx \geq T'(x)x$ for all $x \gg 0$.*

Proof: By continuity it suffices to prove $T|_P$ is sublinear. Fix $\epsilon \gg 0$. For each $\delta > 0$ the map $P \rightarrow Y$, $x \mapsto Tx + \delta\epsilon$ is strongly sublinear by Lemma 6.1. Sending δ to zero implies T is sublinear. \blacksquare

Krause and Ranft [46] have results establishing sublinearity of some iterate of T , which is an assumption used in Theorem 6.3 below.

The following theorem demonstrates global convergence properties for order compact maps that are monotone and sublinear in a suitably strong sense.

Theorem 6.3 (CONE LIMIT SET TRICHOTOMY) *Assume $T : Y_+ \rightarrow Y_+$ is continuous and monotone and has the following properties for some $r \geq 1$:*

- (a) T^r is strongly sublinear
- (b) $T^r x \gg 0$ for all $x > 0$
- (c) T^r is order compact

Then precisely one of the following holds:

- (i) each nonzero orbit is order unbounded
- (ii) each orbit converges to 0, the unique fixed point of T .
- (iii) each nonzero orbit converges to $q \gg 0$, the unique nonzero fixed point of T .

A key tool in the proofs of such results is Hilbert's projective metric and the related part metric due to Thompson [89]. We define the part metric $\mathfrak{p}(x, y)$ here in a very limited way, as a metric on P (which is the "part"). For $x, y \gg 0$, define

$$\mathfrak{p}(x, y) := \inf\{\rho > 0 : e^{-\rho}x \ll y \ll e^{\rho}y\}$$

The family of open order intervals in P forms a base for the topology of the part metric. It is easy to see that the identity map of P is continuous from the original topology on P to that defined by the part metric.

When $Y = \mathbb{R}^n$ with vector ordering, with $P = \text{Int}(\mathbb{R}_+^n)$, the part metric is isometric to the max metric on \mathbb{R}^n , defined by $d_{\max}(x, y) = \max_i |x_i - y_i|$, via the homeomorphism $\text{Int}(\mathbb{R}_+^n) \approx \mathbb{R}^n$, $x \mapsto (\log x_1, \dots, \log x_n)$. Restricted to compact sets in $\text{Int}(\mathbb{R}_+^n)$, the part metric and the max metric are equivalent in the sense that there exist $\alpha, \beta > 0$ such that $\alpha\mathfrak{p}(x, y) \leq d_{\max}(x, y) \leq \beta\mathfrak{p}(x, y)$.

The usefulness of the part metric in dynamics stems from the following result. Recall map T between metric spaces is a *contraction* if it has a Lipschitz constant < 1 , and it is *nonexpansive* if it has Lipschitz constant 1. We say T is *strictly nonexpansive* if $\mathfrak{p}(Tx, Ty) < \mathfrak{p}(x, y)$ whenever $x \neq y$.

Proposition 6.4 *Let $T : P \rightarrow P$ be a continuous, monotone, sublinear map.*

- (i) *T is nonexpansive for the part metric.*
- (ii) *If T is strongly sublinear, T is strictly nonexpansive for the part metric.*
- (iii) *If T is strongly monotone, $A \subset P$, and no two points of A are linearly dependent, then $T|_A$ is strictly nonexpansive for the part metric.*
- (iv) *Assume T is strongly sublinear or strongly monotone and let A be as in (iii). If $L \subset A$ is compact (in the norm topology) and $T(L) \subset L$, then the set $L_\infty = \bigcap_{n>0} T^n(L)$ is a singleton.*

Proof: Fix distinct points $x, y \in P$ and set $e^{\mathfrak{p}(x,y)} = \lambda > 1$, so that $\lambda^{-1}x \leq y \leq \lambda x$ and λ is the smallest number with this property. By sublinearity and monotonicity,

$$\lambda^{-1}Tx \leq T(\lambda^{-1}x) \leq Ty \leq T(\lambda x) \leq \lambda Tx \quad (5)$$

which implies $\mathfrak{p}(Tx, Ty) \leq \mathfrak{p}(x, y)$.

If T is strongly sublinear, the first and last inequalities in (5) can be replaced by \ll , which implies $\mathfrak{p}(Tx, Ty) < \mathfrak{p}(x, y)$.

When x and y are linearly independent, $\lambda^{-1}x < y < \lambda x$. If also T is strongly monotone, (5) is strengthened to

$$\lambda^{-1}Tx \leq T(\lambda^{-1}x) \ll Ty \ll T(\lambda x) \leq \lambda Tx$$

which also implies $\mathfrak{p}(Tx, Ty) < \mathfrak{p}(x, y)$.

To prove (iv), observe first that if L is compact in the norm metric, it is also compact in the part metric. In both (ii) and (iii) T reduces the diameter in the part metric of every compact subset of L . Since T maps L_∞ onto itself but reduces its part metric diameter, (iv) follows. ■

Proof of the Cone Limit Set Trichotomy 6.3. We first work under the assumption that $r = 1$. In this case Proposition 6.4 shows that every compact invariant set in P reduces to a fixed point, and there is at most one fixed point in P . It suffices to consider the orbits of points $x \in P$, by (b).

Suppose there is a fixed point $q \gg 0$. There exist numbers $0 < \lambda < 1 < \mu$ such that $x \in [\lambda q, \mu q] \subset P$. For all n we have

$$0 \ll \lambda q = \lambda T^n q \leq T^n(\lambda q) \leq T^n x \leq T^n(\mu q) \leq \mu T^n q = \mu q$$

Hence $O(x) \subset [\lambda q, \mu q]$, so $O(Tx)$ lies in $T([\lambda q, \mu q])$, which is precompact by (c). Therefore $\omega(x)$ is a compact unordered invariant set in P . Proposition 6.4(iii) implies that $\omega(x) = \{q\}$. This verifies (iii).

Case I: If some orbit $O(y)$ is order unbounded, we prove (i). We may assume $y \gg 0$. There exists $0 < \gamma < 1$ such that $\gamma y \ll x$. Then $\gamma T^n y \leq T^n(\gamma y) \leq T^n x$, implying $O(x)$ is unbounded.

Case II: If $0 \in \omega(y)$ for some y , we prove (ii). We may assume $y \gg 0$. Fix $\mu > 1$ with $x \ll \mu y$. Then $0 \leq T^n x \leq T^n(\mu y) \leq \mu T^n y \rightarrow 0$. Therefore $O(x)$ is compact and $T^n x \rightarrow 0$.

Case III: If the orbit closure $\overline{O(x)} \subset [a, b] \subset P$, then (iii) holds. For $\overline{O(x)}$ is compact by (c), so $\omega(x)$ is a nonempty compact invariant set. Because $\omega(x) \subset \overline{O(x)} \subset P$, Case I implies (iii).

Cases I, II and III cover all possibilities, so the proof for $r = 1$ is complete. Now assume $r > 1$. One of the statements (i), (ii) (iii) is valid for T^r in place of T . If (i) holds for T^r , it obviously holds for T . Assume (ii) holds for T^r . If $x > 0$ then $\omega(x) = \{0, T(0), \dots, T^{r-1}(0)\}$. As this set is compact and T^r invariant, it reduces to $\{0\}$, verifying (ii) for T . A similar argument shows that if (iii) holds for T^r , it also holds for T . ■

The conclusion of the Cone Limit Trichotomy can fail for strongly monotone sublinear maps— simple linear examples in the plane have a line of fixed points. But the following holds:

Theorem 6.5 *Assume:*

- (a) $T : Y_+ \rightarrow Y_+$ is continuous, sublinear, strongly monotone, and order compact.
- (b) for each $x > 0$ there exists $r \in \mathbb{N}$ such that $T^r x \gg 0$

Then:

- (i) either $O(x)$ is not order bounded for all $x > 0$, or $O(x)$ converges to a fixed point for all $x \geq 0$;
- (ii) the set of fixed points > 0 has the form $\{\lambda e : a \leq \lambda \leq b\}$ where $e \gg 0$ and $0 \leq a \leq b \leq \infty$.

Proof: Let $y > 0$ be arbitrary. If $O(y)$ is not order bounded, or $0 \in \omega(y)$, the proof of (i) follows Cases I and II in the proof of the Cone Limit Set Trichotomy 6.3. If $\overline{O(y)} \subset [a, b] \subset P$, then $\omega(y)$ is a compact invariant set in P , as in case III of Theorem 6.3. As $\omega(y)$ is unordered, every pair of its elements are linearly independent. Therefore Proposition 6.4(iv) implies $\omega(y)$ reduces to a fixed point, proving (i). The same reference shows that all fixed points lie on a ray $R \subset Y_+$ through the origin, which must pass through some $e \gg 0$ by (b). Suppose p, q are distinct fixed points and $0 \ll p \ll x \ll q$. There exist unique numbers $0 < \mu < 1 < \nu$ such that $x = \mu p = \nu q$. Then

$$Tx \geq \mu Tp = \mu p = x, \quad Tx \leq \nu Tq = \nu q = x$$

proving $Tx = x$. This implies (ii). ■

Papers related to sublinear dynamics and the part metric include Dafermos and Slemrod [16], Krause and Ranft [46], Krause and Nussbaum [47], Nussbaum [60, 61], Smith [71], and Takáč [80, 86]. For interesting applications of sublinear dynamics to higher order elliptic equations, see Fleckinger and Takáč [21, 22].

References

- [1] E.J.Allen, L.J.S.Allen, X.Gilliam, Dispersal and competition models for plants, *J. Math. Biol.* 34(1996) 455-481.
- [2] A. Alexandroff and H. Hopf, *Topologie*, Berlin: Springer (1935).
- [3] N. Alikakos, P. Hess, H. Matano, Discrete order-preserving semigroups and stability for periodic parabolic differential equations, *J. Diff. Eqns.* 82 (1989), 322-341.
- [4] H. Amann, On the existence of positive solutions of nonlinear elliptic boundary value problems, *Indiana U. Math. J.* 21 (1971), 125-146.
- [5] H. Amann, On the number of solutions of nonlinear equations in ordered Banach spaces, *J. Functional Analysis* 11 (1972), 346-384.
- [6] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach Spaces, *SIAM Rev.* 18 (1976), 620-709.
- [7] G. Birkhoff, Extensions of Jentzsch's theorem. *Trans. Amer. Math. Soc.* 85 (1957), 219-227.
- [8] P. Brunovsky, P. Poláčik, and B. Sandstede, Convergence in general periodic parabolic equations in one space variable, *Nonlinear Analysis TMA* 18 (1992) 209-215.
- [9] X. -Y. Chen and H. Matano, Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations, *J. Diff. Eqns.* 78 (1989), 160-190.
- [10] M. Chen, X. Chen and J. K. Hale, Structural stability for time-periodic one-dimensional parabolic equations, *J. Diff. Eqns.* 96 (1992), 355-418
- [11] R. Courant and D. Hilbert *Methoden der mathematischen Physik*, Bd. 2. Berlin: Verlag Julius Springer (1937).
- [12] J.M. Cushing, Two species competition in a periodic environment, *J.Math. Biol.* 10(1980), 385-400.
- [13] J.M. Cushing, Periodic Lotka-Volterra competition equations, *J.Math. Biol.* 24(1986), 381-403.
- [14] J.M. Cushing, Nonlinear matrix models and population dynamics, *Natural Resource Modeling*, 2(1988) 539-580.
- [15] J. Cushing, S. Leverage, N. Chitnis, S. Henson, Some discrete competition models and the competitive exclusion principle, *J. Difference Eqns. & Appl.*, in press (2004).
- [16] C. M. Dafermos and M. Slemrod, Asymptotic behaviour of nonlinear contractions semigroups, *J. Functional Analysis* 13 (1973), 97-106

- [17] E. Dancer and P. Hess, Stability of fixed points for order-preserving discrete-time dynamical systems, *J. Reine Angew. Math.* 419 (1991), 125-139.
- [18] E. Dancer, Some remarks on a boundedness assumption for monotone dynamical systems, *Proc. Amer. Math. Soc.* 126 (1998), 801-807.
- [19] K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, 1985.
- [20] P. de Mottoni and A. Schiaffino, Competition systems with periodic coefficients: a geometric approach, *J. Math. Biology* 11 (1982), 319-335
- [21] J. Fleckinger-Pellé and P. Takáč, Uniqueness of positive solutions for nonlinear cooperative systems with the p -Laplacian, *Indiana Univ. Math. J.* 43 (1994), 1227-1253
- [22] J. Fleckinger-Pellé and P. Takáč, Unicité de la solution d'un système non linéaire strictement coopératif. *C. R. Acad. Sci. Paris Sér. I Math.* 319 (1994), 447-450.
- [23] G. Frobenius, Über aus nicht negativen Elementen, *Sitzungber. Königl. Preuss. Akad. Wiss.* (1912), 456-477
- [24] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence (1988).
- [25] J. K. Hale, Dynamics of a scalar parabolic equation, *Canadian Applied Math. J.*, 5 (1997), 209-306.
- [26] J. K. Hale and A. S. Somolinos, Competition for fluctuating nutrient, *J. Math. Biol.* 18 (1983), 255-280.
- [27] D.R.Hart and R.H.Gardner, A spatial model for the spread of invading organisms subject to competition, *J. Math. Biol.* 35(1997), 935-948.
- [28] P. Hess, *Periodic-parabolic Boundary Value Problems and Positivity*, Longman Scientific and Technical, New York (1991).
- [29] P. Hess and A. C. Lazer, On an abstract competition model and applications, *Nonlinear Analysis T. M. A.* 16 (1991), 917-940.
- [30] P. Hess and P. Poláčik, Boundedness of prime periods of stable cycles and convergence to fixed points in discrete monotone dynamical systems, *SIAM J. Math. Anal.* 24 (1993) 1312-1330.
- [31] M. W. Hirsch, Systems of differential equations which are competitive or cooperative. I: limit sets. *SIAM J. Appl. Math.* 13 (1982), 167-179.
- [32] M. W. Hirsch, Attractors for discrete-time monotone dynamical systems in strongly ordered spaces, *Geometry and Topology, Lecture Notes in Mathematics* vol. 1167, J. Alexander and J. Harer eds., 141-153. New York: Springer-Verlag (1985).
- [33] M. W. Hirsch, Stability and Convergence in Strongly Monotone dynamical systems, *J. reine angew. Math.* 383 (1988), 1-53.

- [34] M.W. Hirsch and H.L. Smith, Monotone Dynamical Systems, preprint, 2004.
- [35] V. Hutson, K. Mischaikow, P. Poláčik, The evolution of dispersal rates in a heterogeneous time-periodic environment, *J. Math. Biol.* 42 (2001) 501-533.
- [36] S.-B. Hsu, H. Smith, and P. Waltman, Competitive exclusion and coexistence for competitive systems on ordered banach spaces, *Trans. Amer. Math. Soc.* 348 (1996) 4083-4094.
- [37] R. Jentzsch, Uber Integralgleichungen mit positiven Kern, *J. Reine Angew. Math.* 141 (1912), 235–244.
- [38] J.-F. Jiang & S.-X. Yu, *Stable cycles for attractors of strongly monotone discrete-time dynamical systems*, *J. Math. Anal. Appl.* **202** (1996), 349-362.
- [39] H.B. Keller and D.S. Cohen, Some positive problems suggested by nonlinear heat generation, *J. Math. Mech.* 16 (1967), 327-342.
- [40] H.B. Keller, Elliptic boundary value problems suggested by nonlinear diffusion processes, *Arch. Rat. Mech. Anal* 35 (1969), 363-381.
- [41] H.B. Keller, Positive solutions of some nonlinear eigenvalue problems, *J. Math. Mech.* 19 (1969), 279-296.
- [42] M. Krasnoselskii, Positive solutions of operator equations, Groningen, Noordhoff (1964).
- [43] M. Krasnoselskii, The operator of translation along trajectories of differential equations, *Transl. Math. Monographs* 19, Amer. Math. Soc. Providence (1968).
- [44] M.A. Krasnosel'skii and L. Ladyshenskaya, The structure of the spectrum of positive nonhomogeneous operators, *Trudy Moskov. Mat. Obšč.* 3 (1954), 321-346 [Russian].
- [45] M.A. Krasnosel'skii and P.P. Zabreiko, Geometric methods of nonlinear analysis, New York: Springer-Verlag (1984).
- [46] U. Krause and P. Ranft, A Limit Set Trichotomy for monotone nonlinear dynamical systems, *Nonlinear Analysis, TMA* 19 (1992), 375-392.
- [47] U. Krause and R. D. Nussbaum, A Limit Set Trichotomy for self-mappings of normal cones in Banach spaces, *Nonlinear Analysis, TMA* 20 (1993), 855-870.
- [48] M. G. Krein and M. A. Rutman, Linear Operators leaving invariant a cone in a Banach space, *Uspekhi Mat. Nauk* 3 (1948), 3-95. A. M. S. Translation No. 26, Amer. Math. Soc., Providence, R. I., 1950.
- [49] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, 2nd ed., Springer 1998.
- [50] J. P. LaSalle, *The Stability of Dynamical Systems*, SIAM, Philadelphia, PA 1976.

- [51] A. Lasota and M.C. Mackey, Probabilistic properties of deterministic systems, Cambridge Univ. Press, 1985.
- [52] B. Lemmens, R. D. Nussbaum, S. M. Verduyn Lunel, Lower and upper bounds for ω -limit sets of nonexpansive maps. *Indag. Math. (N.S.)* 12 (2001), 191–211.
- [53] P.H.Leslie, On the use of matrices in certain populations, *Biometrika* 33 (1945) 183-212.
- [54] P. Leslie and J. Gower, The properties of a stochastic model for two competing species, *Biometrika* 45 (1958), 316-330.
- [55] X. Liang and J. Jiang, On the finite-dimensional dynamical systems with limited competition, *Trans. Amer. Math. Soc.* 354 (2002), 3535-3554.
- [56] R. Lui, A nonlinear integral operator arising from a model in population genetics IV. Clines, *SIAM J. Math. Anal.* 17 (1986), 152-168.
- [57] H. Matano, Existence of nontrivial unstable sets for equilibriums of strongly order preserving systems, *J. Fac. Sci. Univ. Kyoto* 30 (1984), 645-673.
- [58] H. Matano, Strong Comparison Principle in Nonlinear Parabolic Equations, in *Nonlinear Parabolic Equations: Qualitative Properties of Solutions*, L. Boccardo and A. Tesei, eds., Pitman Research Notes in Mathematics, Longman Scientific and Technical, London, 148-155 (1987).
- [59] J. Mierczyński, P-arcs in strongly monotone discrete-time dynamical systems, *Differential Integral Equations* 7 (1994), 1473-1494.
- [60] R. D. Nussbaum, Hilbert’s projective metric and iterated nonlinear maps, *Memoirs of the Amer. Math. Soc.* 391 (1988).
- [61] R. D. Nussbaum, Iterated nonlinear maps and Hilbert’s projective metric, II, *Memoirs of the Amer. Math. Soc.* 401 (1989).
- [62] R. D. Nussbaum, Estimates of the periods of periodic points for nonexpansive operators. *Israel J. Math.* 76 (1991), 345–380.
- [63] O. Perron, Zur Theorie der Matrizen, *Math. Annalen* 64 (1907), 248-263
- [64] P. Poláčik, Existence of unstable sets for invariant sets in compact semiflows. Applications in order-preserving semiflows, *Comment. Math. Univ. Carolinae* 31 (1990), 263-276.
- [65] P. Poláčik, Parabolic equations: asymptotic behavior and dynamics on invariant manifolds. *Handbook of Dynamical Systems Vol. 2* (B. Fiedler, editor). Elsevier, New York 2002.
- [66] P. Poláčik and I. Tereščák, Convergence to cycles as a typical asymptotic behavior in smooth strongly monotone discrete-time dynamical systems. *Arch. Rational Mech. Anal.* 116 (1992), 339-360.

- [67] P. Poláčik and I. Tereščák, Exponential separation and invariant bundles for maps in ordered Banach spaces with applications to parabolic equations, *J. Dynam. Differential Equations* 5 (1993), 279-303
- [68] M. Rutman, Sur les opérateurs totalement continus linéaires laissant invariant un certain cône, *Rec. Math. [Mat. Sbornik] N. S.* 8 (50), (1940), 77–96
- [69] Sattinger, D. (1972) Monotone Methods in nonlinear elliptic and parabolic boundary value problems, *Indiana Univ. Math. J.* 21, 979-1000.
- [70] J. Selgrade and M. Ziehe, Convergence to equilibrium in a genetic model with differential viability between sexes, *J. Math. Biol.* 25 (1987), 477-490.
- [71] H. L. Smith, Cooperative systems of differential equations with concave nonlinearities, *Nonlinear Analysis* 10 (1986), 1037-1052.
- [72] H. L. Smith, Invariant curves for mappings, *SIAM J. Math. Anal.* 17 (1986), 1053-1067.
- [73] H. L. Smith, Periodic competitive differential equations and the discrete dynamics of competitive maps, *J. Diff. Eqns.* 64 (1986), 165-194.
- [74] H. L. Smith, Periodic solutions of periodic competitive and cooperative systems, *Siam J. Math. Anal.* 17 (1986), 1289-1318.
- [75] H. L. Smith, *Monotone Dynamical Systems, an introduction to the theory of competitive and cooperative systems*, Math. Surveys and Monographs, 41, American Mathematical Society, Providence, Rhode Island (1995).
- [76] H. L. Smith, Complicated dynamics for low-dimensional strongly monotone maps, *Nonlinear Analysis TMA* 30 (1997) 1911-1914.
- [77] H. L. Smith, Planar competitive and cooperative difference equations, *J. Difference Eqns. and Appl.* 3 (1998) 335-357.
- [78] H. L. Smith and H. R. Thieme, Quasi convergence for strongly ordered preserving semiflows, *SIAM J. Math. Anal.*, 21 (1990), 673-692.
- [79] H. L. Smith and H. R. Thieme, Stable coexistence and bi-stability for competitive systems on ordered Banach spaces, *J. Diff. Eqns.* 176 (2001), 195-222.
- [80] P. Takáč, Asymptotic behavior of discrete-time semigroups of sublinear, strongly increasing mappings with applications in biology, *Nonlinear Analysis* 14 (1990), 35-42.
- [81] P. Takáč, Convergence to equilibrium on invariant d-hypersurfaces for strongly increasing discrete-time semigroups, *J. Math. Anal. and Appl.* 148 (1990), 223-244.
- [82] P. Takáč, Domains of attraction of generic ω -limit sets for strongly monotone semiflows, *Zeitschrift fur Analysis und ihre Anwendungen*, 10 (1991), 275-317.

- [83] P. Takáč, Domains of attraction of generic ω -limit sets for strongly monotone discrete-time semigroups, *J. Reine Angew. Math.* 432 (1992) 101-173.
- [84] P. Takáč, Linearly stable subharmonic orbits in strongly monotone time-periodic dynamical systems, *Proc. Amer. Math. Soc.* 115 (1992) 691-698.
- [85] P. Takáč, A short elementary proof of the Kreĭn-Rutman theorem. *Houston J. Math.* 20 (1994), 93-98.
- [86] P. Takáč, Convergence in the part metric for discrete dynamical systems in ordered topological cones, *Nonlinear Analysis TMA*, 26 (1996) 1753-1777.
- [87] P. Takáč, Discrete monotone dynamics and time-periodic competition between two species, *Differential Integral Equations* 10 (1997), 547-576
- [88] I. Tereščák, Dynamics of C^1 smooth strongly monotone discrete-time dynamical system, Preprint, Comenius University, Bratislava, 1994.
- [89] A. C. Thompson, On certain contraction mappings in a partially ordered vector space, *Proc. Amer. Math. Soc.* 14 (1963) 438-443.
- [90] H. R. Thieme, Density dependent regulation of spatially distributed populations and their asymptotic speeds of spread, *J. Math. Biol.* 8 (1979), 173–187.
- [91] P.S. Uryson, Mittlelehre Breite und Volumen der konvexen Körper im n -dimensionalen Räume, *Mat. Sb.* 31 (1924), 477-486.
- [92] Y. Wang and J. Jiang, The general properties of discrete-time competitive dynamical systems. *J. Differential Equations* 176 (2001), 470-493.
- [93] Y. Wang and J. Jiang, Uniqueness and attractivity of the carrying simplex for discrete-time competitive dynamical systems. *J. Differential Equations* 186 (2002), 611-632.
- [94] Y. Wang and J. Jiang, The long-run behavior of periodic competitive Kolmogorov systems. *Nonlinear Anal. Real World Appl.* 3 (2002), 471-485.
- [95] H. F. Weinberger, Long-time behavior of a class of biological models, *SIAM J. Math. Anal.* 13 (1982) 353-396.
- [96] H.F. Weinberger, M.A.Lewis, B. Li, Analysis of the linear conjecture for spread in cooperative models, *J. Math. Biol.* 45(2002) 183 - 218.
- [97] J. Wu, H. I. Freedman, R. K. Miller, Heteroclinic orbits and convergence of order-preserving set-condensing semiflows with applications to integrodifferential equations, *J. Integral Equations Appl.* 7 (1995), 115-133.
- [98] F. Zanolin, Permanence and positive periodic solutions for Kolmogorov competing species systems, *Results in Mathematics*, 21 (1992), 224-250.
- [99] E. Zeidler, *Nonlinear Functional Analysis and its Applications I Fixed Point Theorems*, Springer-Verlag, New York (1986).

- [100] X.-Q. Zhao, Global attractivity and stability in some monotone discrete dynamical systems, *Bull. Austral. Math. Soc.* 53 (1996), 305-324.