Journal of Difference Equations and Applications,
Vol. 00, No. 00, Month 200x, 1–6

Global Stability for Mixed Monotone Systems

H.L. Smith*

Department of Mathematics and Statistics, Arizona State University, Tempe, AZ 85287

(*v3.1 released August 2006*)

We show that the embedding method described in [4,8] leads immediately to the global stability results in [7]. It also allows extension of some results on global stability for higher order difference equations due to Gerry Ladas and collaborators. Further, we provide a new result which suggests that embedding into monotone systems may not be necessary for global stability results.

Keywords: mixed monotone system, monotone dynamical system, global stability

This paper is dedicated to Gerry Ladas on the occasion of his 70th Birthday

1. Introduction

The idea of embedding a dynamical system, whose generator has both increasing and decreasing monotonicity properties (positive and negative feedback), into a larger symmetric monotone dynamical system and exploiting the convergence properties of the latter is very old. For a discussion of history of the method, see [4,8]. The method is repeatedly rediscovered and its implications are often underestimated. In this paper, we review the main results of the embedding method following [8] and then we show how it leads immediately to an improved version of a nice result on global stability due to Kulenović and Merino [7] for componentwise monotone maps that leave invariant a hypercube in Euclidean space. We then ask whether embedding a system into a larger monotone system is really necessary to obtain global stability results. On the face of it, it seems unnatural to pass to a larger dimensional dynamical system in order to gain information on the dynamics of a smaller one. We show that for the class of mixed-monotone systems, one can obtain global stability results directly without the need of embedding.

As noted in [8], the embedding method leads to a nice generalization of some results of Kulenović, Ladas and Sizer [5], also contained in the monograph of Kulenović and Ladas [6], on higher order difference equations with componentwise monotonicity.

2. Review of the Embedding Method

Let X be an ordered metric space with closed order relation $\leq$. The closedness of the order relation means that if $x_n \leq y_n$ and $x_n \to x,\ y_n \to y$, then $x \leq y$. If $x \leq y$, define the order interval $[x, y] := \{z \in X : x \leq z \leq y\}$. Let $F : X \to X$ be continuous. Our focus is the discrete-time dynamical system defined by

$$x' = F(x) \tag{1}$$

where x' denotes the successor to x.

---

*Corresponding author. Email: halsmith@asu.edu

Journal of Difference Equations and Applications
ISSN: 1023-6198 print/ISSN 1563-5120 online
http://www.tandf.co.uk/journals
DOI: 10.1080/1023619YYxxxxxxxx

We say F is mixed-monotone if there exists a continuous map $f : X \times X \to X$ satisfying:

(1) $F(x) = f(x,x),\ x \in X$.

(2) $\forall y \in X,\ x_1, x_2 \in X,\ x_1 \leq x_2 \Rightarrow f(x_1, y) \leq f(x_2, y)$.

(3) $\forall x \in X,\ y_1 \leq y_2 \Rightarrow f(x, y_2) \leq f(x, y_1)$.

In short, f is nondecreasing in its first variable and nonincreasing in its second. Roughly, F is a map that combines both positive and negative feedback. We write $F^n(x)$ for the n-fold composition of F acting on x. The omega limit set of a subset $A \subset X$ is denoted by $\omega_F(A)$ and that of a single point $x \in X$, is denoted by $\omega_F(x)$.

As shown in [4], (1) can be embedded in the symmetric discrete-time dynamical system

$$\begin{aligned} x' &= f(x,y) \\ y' &= f(y,x) \end{aligned} \tag{2}$$

on $X \times X$. We use the notation $z = (x,y)$ and define

$$G(z) = G(x,y) = (f(x,y), f(y,x))$$

G is called the symmetric map. Obviously, the diagonal

$$D = \{(x,x) : x \in X\}$$

is positively invariant under (2) and $G(x,x) = (F(x), F(x))$. The symmetry of G can be expressed by defining the reflection operator $R(x,y) = (y,x)$ and observing that $G \circ R = R \circ G$.

The "southeast" ordering on $X^2 := X \times X$ is the closed partial order relation defined by

$$(x,y) \leq_C (\bar{x}, \bar{y}) \Longleftrightarrow x \leq \bar{x} \text{ and } \bar{y} \leq y.$$

It's name derives from the fact that the bigger point lies southeast of the smaller one. Note that $R : X^2 \to X^2$ is order reversing.

Although the map F need not be monotone, the symmetric map G is monotone.

Lemma 2.1: *G is monotone with respect to $\leq_C$ on $X \times X$in the sense that*

$$z \leq_C \bar{z} \Longrightarrow G(z) \leq_C G(\bar{z}).$$

Moreover, the "above-diagonal set" $\{(x,y) \in X \times X : x \leq y\}$ is positively invariant under G.

The following result, proved in [8] (see also [2]), is a sharpened version of Theorem 7 in [4].

Theorem 2.2: *Suppose that:*

$$\exists x_0, y_0,\ x_0 \leq y_0,\ \textit{satisfying } f(x_0, y_0) \geq x_0,\ f(y_0, x_0) \leq y_0. \tag{3}$$

Then $F([x_0, y_0]) \subset [x_0, y_0]$ and for $z_0 = (x_0, y_0)$ and $n \geq 1$, we have:

$$z_0 \leq_C G^n(z_0) \leq_C G^{n+1}(z_0) \leq_C R z_0.$$

Assume, in addition, that the monotone orbit

$$\{G^n(x_0, y_0)\}_{n \geq 1} \textit{ converges in } X. \tag{4}$$

Then:

(i) there exist $x_*, y_* \in [x_0, y_0]$ *with* $x_* \leq y_*$ *satisfying*

$$G^n(x_0, y_0) \longrightarrow (x_*, y_*) = G(x_*, y_*),$$

implying that $f(x_*, y_*) = x_*,\ f(y_*, x_*) = y_*$.

(ii) If $x \in [x_0, y_0]$ *and* $\{F^n(x)\}_{n\geq 1}$ *has compact closure in* X*, then*

$$\omega_F(x) \subset [x_*, y_*]. \tag{5}$$

(iii) If $F([x_0, y_0])$ *has compact closure in* X*, then* $\omega_F([x_0, y_0]) \neq \emptyset$*, and*

$$\omega_F([x_0, y_0]) \subset [x_*, y_*]. \tag{6}$$

(iv) If

$$a, b \in [x_0, y_0],\ \ a \leq b,\ \ f(a, b) = a,\ \ b = f(b, a) \Rightarrow a = b \tag{7}$$

holds then $x_* = y_*$ *and* $F(x_*) = x_*$*. In this case, if* $x \in [x_0, y_0]$ *and* $\{F^n(x)\}_{n\geq 1}$ *has compact closure in* X*, then* $\omega_F(x) = \{x_*\}$*.*

As noted in [8], the hypothesis (4) may be satisfied under a variety of hypotheses on either the space X or the map f. If X is a finite dimensional ordered Banach space or if $X = L^p(\Omega)$ is a Lebesgue space then order bounded sequences converge. It holds if f has compactness properties. See [8].

The proof of Theorem 2.2 exploits the fact that (3) is equivalent to:

$$(x_0, y_0) \leq_C G(x_0, y_0)$$

and so the monotone symmetric map G has a monotone increasing (relative to $\leq_C$) orbit $(x_n, y_n) = G^n(x_0, y_0)$ which must remain in the "above-diagonal" set $\{(x, y) \in X \times X : x \leq y\}$ by Lemma 2.1. By monotonicity of G and the fact that $G = (F, F)$ on the diagonal, one concludes that for $x \in [x_0, y_0]$ we have

$$x_n \leq F^n(x) \leq y_n$$

where $x_n \nearrow x_*$ and $y_n \searrow y_*$.

Applications of Theorem 2.2 can be found in [8]. See also those in [7].

3. A result of Kulenović and Merino

We show that the main result of Kulenović and Merino [7] follows from Theorem 2.2.

Let $m, M \in \mathrm{R}^{\mathrm{k}}$ satisfy $m \leq M$ and let $F : [m, M] \to [m, M]$ be a continuous map. Kulenović and Merino call F coordinate-wise monotone (cw-monotone) if F_i is monotone in x_j on $[m_j, M_j]$ for $1 \leq i, j \leq k$. For each i, $K = \{1, 2, \cdots, k\}$ can be partitioned into two disjoint subsets as follows:

$$I_i = \{j \in K : F_i \text{ is nondecreasing or constant in } x_j\}$$

and

$$D_i = \{j \in K : F_i \text{ is nonincreasing and nonconstant in } x_j\}$$

Given vectors $x, y \in [m, M]$ denote by (x_{I_i}, y_{D_i}) the vector $z \in rmR^k$ with $z_l = x_l$ for $l \in I_i$ and $z_l = y_l$ for $l \in D_i$. Observe that $z \in [m, M]$. Define $f : [m, M] \times [m, M] \to \mathrm{R^k}$ by

$$f_i(x, y) = F(x_{I_i}, y_{D_i})$$

Kulenović and Merino give an alternate but equivalent definition of f as follows. Define

$$\sigma_{ij} = \left\{ \begin{array}{l} 1 \quad\;\; \text{if } F_i \text{ is nondecreasing or constant in } x_j \\ -1 \text{ if } F_i \text{ is nonincreasing and nonconstant in } x_j \end{array} \right\}$$

Then for $1 \leq i \leq k$,

$$f_i(x, y) = F_i \left(\frac{1 + \sigma_{i1}}{2} x_1 + \frac{1 - \sigma_{i1}}{2} y_1, \frac{1 + \sigma_{i2}}{2} x_2 + \frac{1 - \sigma_{i2}}{2} y_2, \cdots, \frac{1 + \sigma_{ik}}{2} x_k + \frac{1 - \sigma_{ik}}{2} y_k \right)$$

Observe that

(a) $f([m, M] \times [m, M]) \subset [m, M]$.
(b) $F(x) = f(x, x),\ x \in [m, M]$.
(c) f is nondecreasing in x and nonincreasing in y.

The following result extends Theorem 3 of Kulenović and Merino [7].

Theorem 3.1: *Let $F : [m, M] \to [m, M]$ be a continuous cw-monotone map. Assume that*

$$a, b \in [m, M],\ a \leq b,\ f(a, b) = a,\ b = f(b, a) \Rightarrow a = b \tag{8}$$

holds. Then there exists $x_* \in [m, M]$ such that $\omega_F(x) = x_*$ for all $x \in [m, M]$.

Proof: Since $f : [m, M] \times [m, M] \to [m, M]$, it follows that $f(m, M) \geq m$ and $f(M, m) \leq M$. Hence hypothesis (3) of Theorem 2.2 holds with $x_0 = m$ and $y_0 = M$. Indeed, all hypotheses of that theorem hold so the result follows from part (iv). □

Theorem 3.1 extends Theorem 3 in [7] because the additional restriction $a \leq b$ appears in our premise (8) but not in the premise of II. of Theorem 3 of [7].

4. To Embed or not to Embed

One might wonder whether it is really necessary to embed (1) into the symmetric map (2) in order to obtain significant results. We begin by showing that a seemingly more powerful hypothesis than (8) is actually equivalent to it.

Proposition 4.1: *Suppose that hypotheses (3) and (4) of Theorem 2.2 hold. Then hypothesis (8) holds if and only if*

$$a, b \in [x_0, y_0],\ a \leq b,\ f(a, b) \leq a,\ b \leq f(b, a) \Rightarrow a = b \tag{9}$$

holds.

Proof: Suppose that (8) holds and that there exists $a, b \in [x_0, y_0]$ such that $a \leq b,\ f(a, b) \leq a,\ b \leq f(b, a)$. In terms of the symmetric map G, this means that $G(a, b) \leq_C (a, b)$. Then $(x_0, y_0) \leq_C (a, b)$ and so by monotonicity of G

$$(x_0, y_0) \leq_C G(x_0, y_0) \leq_C G(a, b) \leq_C (a, b).$$

It follows that $G(x_*, y_*) = (x_*, y_*) = \lim_n G^n(x_0, y_0) \leq_C (a, b)$. So $f(x_*, y_*) = x_*$, $f(y_*, x_*) = y_*$ and $x_* \leq y_*$. (8) implies that $x_* = y_*$. The inequality $(x_*, y_*) \leq_C (a, b)$ implies that $b \leq y_* = x_* \leq a$ which, together with $a \leq b$ implies $a = b$. □

Now suppose that X is a nonempty subset of an ordered metric space Z. We say that the continuous map $F : X \to X$ is weakly mixed-monotone if there exists a (not necessarily continuous) map $f : X \times X \to Z$ (note the range space is Z!) satisfying:

(1) $F(x) = f(x, x),\ x \in X$.
(2) $\forall y \in X,\ x_1, x_2 \in X,\ x_1 \leq x_2 \Rightarrow f(x_1, y) \leq f(x_2, y)$.

(3) $\forall x \in X,\ y_1 \leq y_2 \Rightarrow f(x, y_2) \leq f(x, y_1)$.

Theorem 4.2: *Suppose that F is weakly mixed monotone and:*

$$a, b \in X,\ a \leq b,\ f(a, b) \leq a,\ b \leq f(b, a)\ \Rightarrow a = b. \tag{10}$$

If X contains the infimum and supremum of any pair of fixed points of F, then F has at most one fixed point in X.

If $x \in X$ has the property that $\overline{\{F^t(x)\}_{t=0}^{t=\infty}}$ is compact and contained in X and such that $\inf \omega_F(x)$ and $\sup \omega_F(x)$ exist in Z and belong to X, then $\omega_F(x)$ is a fixed point of F.

Proof: First, observe that (10) implies uniqueness of fixed points for F in X. If $f(a, a) = a$ and $f(b, b) = b$ for $a, b \in X$, set $A = \inf\{a, b\}$ and $B = \sup\{a, b\}$. Then $A, B \in X$ by hypothesis and $A \leq a, b \leq B$. By the weak mixed monotone condition, $f(A, B) \leq f(a, a) = a$ and $f(A, B) \leq f(b, b) = b$ so $f(A, B) \leq A$. Similarly, $f(B, A) \geq B$. Therefore, $A = B$ and $a = b$ by (10).

Suppose that $x \in X$ has the property that $\overline{\{F^t(x)\}_{t=0}^{t=\infty}}$ is compact and contained in X and such that $c = \inf \omega_F(x)$ and $d = \sup \omega_F(x)$ exist in Z and belong to X. Therefore $\omega_F(x)$ is nonempty and invariant: $F(\omega_F(x)) = \omega_F(x)$. If $y \in \omega_F(x)$, there exists $z \in \omega_F(x)$ such that $F(z) = y$. By the properties of f and the fact that $c \leq z \leq d$, we conclude that $f(c, d) \leq y = F(z) = f(z, z) \leq f(d, c)$. Therefore, $f(c, d) \leq \omega(x) \leq f(d, c)$ which implies that $f(c, d) \leq c$ and $f(d, c) \geq d$. (10) implies that $c = d$. Therefore, $\omega_F(x)$ is a singleton, necessarily a fixed point. □

In the special case that $X = [x_0, y_0] \subset \mathrm{R}^\mathrm{k}$ with any orthant ordering or $X = [x_0, y_0] \subset C(A, \mathrm{R}^\mathrm{k})$ where A is a compact space and $C(A, \mathrm{R}^\mathrm{k})$ is ordered in the natural way from some orthant ordering on R^k, and if orbits of F are precompact, then the second part of Theorem 4.2 extends Theorem2.2 (iv). Indeed, compact subsets of $C(A, \mathrm{R}^\mathrm{k})$ have infima and suprema.

Observe from the proof of Theorem 4.2 that the map f is used in a very limited way compared to its use in Theorem2.2. Unlike its use in the proof of Theorem2.2 where it defines the map G, in the proof of Theorem 4.2 there is no need to iterate f, nor is its continuity required. These facts allow weakening the hypotheses on f for Theorem 4.2.

References

[1] C. Cosner, Comparison Principles for systems that embed in cooperative systems, with applications to diffusive Lotka-Volterra Models, Dynam. Continuous, Discrete, and Impulsive Systems, 3 (1997), 283-303.
[2] G. Enciso, E. Sontag and H.L. Smith, Non-monotone systems decomposable into monotone systems with negative feedback, Journal of Differential Equations, Volume 224, Issue 1, 1 May 2006, 205-227.
[3] J.-L. Gouzé, A criterion of global convergence to equilibrium for differential systems with an application to Lotka-Volterra systems, Rapport de Recherche 894, 1988 INRIA.
[4] J.-L. Gouzé and K.P. Hadeler, Monotone Flows and order intervals, Nonlinear World 1, 1994, 23-34.
[5] M. Kulenović and G. Ladas, Dynamics of second order rational difference equations, Chapman & Hall/CRC, Boca Raton Fl. 2002.

[6] M. Kulenović, G. Ladas, W.Sizer, On the recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1})/(\gamma x_n + \delta x_{n-1})$, Math. Sci. Res. Hot-Line 2(5) 1998,1-16.

[7] M. Kulenović and O. Merino, A global attractivity result for maps with invariant boxes, Discrete and Cont. Dynamical Systems, ser.B, 6 (2006) 97-110.

[8] H.L. Smith, The discrete dynamics of monotonically decomposable maps, J. Math. Biol. 53, 2006, 747-758.

99