Stable Coexistence and Bi-stability for Competitive Systems on Ordered Banach Spaces

by

H.L. Smith
Department of Mathematics
Arizona State University
Tempe, AZ 85287–1804

H.R. Thieme
Department of Mathematics
Arizona State University
Tempe, AZ 85287-1804

This paper is dedicated to Willi Jäger on the occasion of his 60$^{\text{th}}$ birthday.

ABSTRACT. Autonomous and asymptotically autonomous semiflows modeling two species competition are studied which are strongly order preserving on a convex subset of a Banach space with competitive order. Conditions are derived for stable coexistence, bi-stability, and competitive exclusion. A complete classification of all possible outcomes is obtained in case that there is at most one positive steady state representing coexistence of both species.

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0. Introduction

Mathematical models of competition between two populations give rise to equations of great variety. Models in which space plays no role may take the form of ordinary differential equations, or of delay differential equations. These equations may be nonautonomous if seasonal or diurnal periodicities in the environment are important. Models for which space plays an important role may take the form of advection-diffusion systems, possibly including time delays or time-varying coefficients. In some cases, discrete time systems arise directly from the model. See [Sm5] for a review, the references of [Sm5] and [CJ1, CJ2, CHL, HsW, HSW1, Sm1, Sm6, STW, SW1, SW2] for an admittedly biased sample of models.

Despite this variety of form, the analysis of mathematical models of two-species competition has many common features, as was first noted by Hess and Lazer [HL]. As each population density should be nonnegative, it is natural to take its state space to be the positive cone in a suitable Banach space. Then, for two-species competition, the appropriate state space is the product space of the two cones. The dynamics may be described by a mapping, for discrete time models, or by a semiflow, for continuous time models, on the product space which preserves a natural order relation for competition. Hess and Lazer exploit strong monotonicity, smoothness of the mapping and the Krein-Rutman Theorem, applied to appropriate linearizations, to give an abstract theory of discrete time competitive dynamics. Later work of Hsu, Waltman and Ellermeyer [HWE] considers continuous time competitive systems and uses the theory of monotone dynamical systems. Hsu, Smith and Waltman [HSW2] take a more topological approach, relaxing the smoothness hypotheses used by Hess and Lazer but retaining a strong compactness assumption and the hypothesis that the positive cones have non-empty interior. They show that either there exists a steady state representing coexistence of the populations or competitive exclusion holds. Takáč [Ta2], using the theory of ejective fixed points and strong monotonicity, gives sufficient conditions for all orbits of a discrete-time competitive system initiating from the order interval determined by the two single-population fixed points to converge to a fixed point. In this paper, we focus our attention primarily on the dynamics of competition when one or more coexistence steady states exist. Using the theory of monotone semiflows, especially [ST1, ST2], we are able to reduce the compactness assumption and, in most cases, to drop the hypothesis that the positive cones have interior. For the so-called bi-stable case where a single saddle-point coexistence steady state exists, we show that a "thin" separatrix separates the basins of attraction of the two single-population steady states. See Iida et al [IMN] for a discussion of the separatrix for the diffusive Lotka-Volterra model. If, on the other hand, both single-population steady states are unstable in a weak sense, then we show that two, not necessarily distinct, order-related coexistence steady states exist. These steady states have substantial basins of attraction and, if they coincide, there is a unique coexistence steady state. In the general case, each orbit approaches either one of the single-population steady states or the order interval generated by the two coexistence steady states and the omega limit set of the generic orbit consists of equilibria.

In the applications it frequently occurs that a competitive system arises as a limiting system for some other dynamical system which itself models competition but does not possess the properties considered here. Competition between two microbial strains for a limiting nutrient is a notable example [HsW, HSW1, Sm1, SW1, SW2, STW]. A "conservation law" for total biomass implies that the weighted sum of microbial densities and nutrient density equilibrates. Hence, asymptotically in time, the nutrient equation may be eliminated. A similar situation occurs in a model of competing strains of a sexually transmitted disease [CHL]. Therefore, we are motivated to treat the case of asymptotically autonomous semiflows which are asymptotic as time tends to infinity to a limiting semiflow which has the features of a competitive semiflow. Here, the goal is to determine what dynamical features of the competitive semiflow can be lifted to the asymptotically autonomous one. One such feature, it is shown, is convergence to one of the steady states.

The abstract setup for competition is described in the next section and our main results are previewed. Subsequent sections provide proofs and elaborations. A general discussion of our results is provided in a concluding section where an overview is given in case of at most one coexistence equilibrium.

1. Main Results

In this section we introduce some notation and standing assumptions and preview some of our main results. For $i = 1, 2$, let X_i be ordered Banach spaces with positive cones X_i^+ and denote by $\text{Int}X_i^+$, the interior of X_i^+. We do not generally assume that X_i^+ has nonempty interior. The same symbol for the partial orders generated by the cones X_i^+ are used. If $x_i, \bar{x}_i \in X_i$, then we write $x_i \leq \bar{x}_i$ if $\bar{x}_i - x_i \in X_i^+$, $x_i < \bar{x}_i$ if $x_i \leq \bar{x}_i$ and $x_i \neq \bar{x}_i$, and $x_i \ll \bar{x}_i$ if $\bar{x}_i - x_i \in \text{Int}X_i^+$. If $x_i, y_i \in X_i$ satisfies $x_i < y_i$, then the order interval $[x_i, y_i]$ is defined by $[x_i, y_i] = \{u \in X_i : x_i \leq u \leq y_i\}$. If $x_i \ll y_i$, then $[[x_i, y_i]] = \{u \in X_i : x_i \ll u \ll y_i\}$ is called an open order interval. We use the same notation for the norm in both X_1 and X_2, namely $\| \bullet \|$.

Let $X = X_1 \times X_2$, $X^+ = X_1^+ \times X_2^+$. X^+ is a cone in X with $\text{Int}X^+ = \text{Int}X_1^+ \times \text{Int}X_2^+$ (possibly empty). It generates the order relations $\leq, <, \ll$ in the usual way. In particular, if $x = (x_1, x_2)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2)$, then $x \leq \bar{x}$ if and only if $x_i \leq \bar{x}_i$, for $i = 1, 2$. For our purposes, the more important cone is $K = X_1^+ \times (-X_2^+)$ with (possibly empty) interior given by $\text{Int}K = \text{Int}X_1^+ \times (-\text{Int}X_2^+)$. It generates the partial order relations $\leq_K, <_K, \ll_K$. In this case,

$$x \leq_K \bar{x} \iff x_1 \leq \bar{x}_1 \quad \text{and} \quad \bar{x}_2 \leq x_2.$$

A similar statement holds with $\ll_K$ replacing $\leq_K$ and $\ll$ replacing $\leq$. We consider a closed convex subset C of X^+ and let

$$C_0 = \{x = (x_1, x_2) \in C : \ x_i > 0, i = 1, 2\},$$

$$C_1 = \{(x_1, 0) \in C : x_1 > 0\},$$

$$C_2 = \{(0, x_2) \in C : x_2 > 0\}.$$

If $x <_K y$, then $[x, y]_K = \{z \in C : x \leq_K z \leq_K y\}$. A set L is said to be **unordered** if it does not contain distinct points related by $<_K$. L is said to be **linearly ordered** if $x <_K y$ or $y <_K x$ for any two distinct points x, y of L. An inequality $A \leq_K B$ between two subsets A and B of X means that the indicated inequality holds between any choice of elements from each set. If $x = (x_1, x_2) \in X$, then we define $\|x\| = \|x_1\| + \|x_2\|$.

Assume that $T : [0, \infty) \times C \to C$ is a continuous semiflow. We write $T_t(x) = T_t x = T(t, x)$. The semiflow properties are (i) $T_0(x) = x$ for all $x \in C$, and (ii) $T_t \circ T_s = T_{t+s}$ for $t, s \geq 0$. If $x \in C$ then $O(x) = \{T_t(x) : t \geq 0\}$ is called the positive orbit of T. Its omega limit set, $\omega(x)$, is defined in the usual way. An **equilibrium** is a point x for which $O(x) = \{x\}$. We say that x is a **convergent point** if $\omega(x)$ is a singleton set. It is a **quasi-convergent point** if $\omega(x)$ consists of equilibria. Semiflow T is **order preserving** if $T_t(x) \leq_K T_t(y)$ whenever $x \leq_K y$ and **strictly order preserving** if $T_t(x) <_K T_t(y)$ whenever $x <_K y$. It is **strongly order preserving on** A, for some positively invariant subset $A \subset C$, if it is order preserving and whenever $x, y \in A$ and $x <_K y$, there exist relatively open sets U and V in A, $x \in U$ and $y \in V$, and $t_0 \geq 0$ such that $T_{t_0}(U) \leq_K T_{t_0}(V)$. An equilibrium x_0 is **locally attractive from above(below)** if there is a neighborhood U of x_0 such that $T_t(x) \to x_0$ as $t \to \infty$ for all $x \in U$ with $x_0 \leq_K x$ ($x \leq_K x_0$). It is a **uniform strong repeller** for set V if there exists $\epsilon > 0$ such that $\liminf_{t \to \infty} \|T_t(x) - x_0\| > \epsilon$ for all $x \in V$.

We introduce the following assumptions. (H0)-(H3) will always be assumed to hold, while (H4) and (H5) will occasionally be invoked.

(H0) $E_0 = (0, 0) \in C$. Further, if $(x_1, x_2) \in C$, then $(x_1, 0) \in C$ and $(0, x_2) \in C$.

(H1) T is strictly order preserving on C, $T_tC_0 \subset C_0$, and T is strongly order preserving on C_0. For each $x \in C$, $O(x)$ has compact closure in C.

(H2) The equilibria of T include: $E_0 = (0,0)$, and unique equilibria $E_1 = (\hat{x}_1, 0) \in C_1$, and $E_2 = (0, \hat{x}_2) \in C_2$. E_0 has a trivial basin of attraction: if $\omega(x) = E_0$ then $x = E_0$.

(H3) $T_t(C_1) \subset C_1$ for all $t \geq 0$ and $T_t(x) \to E_1$ as $t \to \infty$ for all $x \in C_1$. Symmetric conditions hold for T on C_2 with globally attracting equilibrium point E_2.

(H4) There exists a unique equilibrium of T in C_0 denoted by $E = (\bar{x}_1, \bar{x}_2) \in C_0$.

(H5) There exist $x, z \in C_1$, $t \geq 0$, and a relatively open set U in C_1 such that $E_1 \in U$ and $x \leq_K T_t(U) \leq_K z$. Symmetric conditions hold for T on C_2 with equilibrium point E_2.

Remark. There are at least two scenarios in which (H5) holds for C_1:

(a) T is strongly order preserving when restricted to the forward invariant set C_1 and there exists some $w \in C_1$ such that $E_1 <_K w$.

(b) $E_1 \in \text{Int } C_1$.

The order interval

$$I \equiv ([0, \hat{x}_1] \times [0, \hat{x}_2]) \cap C = [E_2, E_1]_K$$

plays a distinguished role in the theory. First, it is positively invariant by (H1) and (H2). Furthermore, it attracts the orbits of all points because, for each $x = (x_1, x_2) \in C_0$, $\omega(x) \subset I$. Indeed, from the inequality $(0, x_2) <_K x <_K (x_1, 0)$ we have $T_t(0, x_2) <_K T_t(x) <_K T_t(x_1, 0)$. Letting $t \to \infty$ along a sequence and using (H3) gives the result. In particular, all steady states belong to I. We denote by

$$B_i = \{x : \omega(x) = E_i\}$$

the basin of attraction of E_i, $i = 1, 2$ and, if (H4) holds, let

$$B = \{x : \omega(x) = E\}$$

be the basin of attraction of E.

We first consider the case that E is unstable and E_1 and E_2 are locally attracting. In this case, we expect the existence of a "thin" separatrix bounding the basins of attraction of the E_i.

Theorem 1. *Let (H0)-(H4) hold. Suppose that E is an interior point of C_0 (in particular, X^+ has nonempty interior) and there is a neighborhood U of E in X with $U \subset C_0$ such that T_{t_0} is a C^1 map in U, the spectral radius of $D_xT_{t_0}(E)$ is strictly greater than one for some positive t_0, and the radius of its essential spectrum is strictly less than one. Furthermore, assume that B_i contains a neighborhood of E_i in C for $i = 1, 2$. Let $S = C \setminus (B_1 \cup B_2)$. Then:*

(a) $\{x <_K E\} \subset B_2$; $\{x >_K E\} \subset B_1$.

(b) *S is an unordered, positively invariant set consisting of E_0, E, the basin of attraction B and, possibly, a set of points which are not quasi-convergent.*

(c) *$B_1 \cup B_2$ is open and dense in C and, if X is finite dimensional, then the Lebesgue measure of S is zero.*

Remark 1: The set S is "thin". Let $w \in K$ be nonzero and $v \in C$. As S is unordered, there is at most one value of λ such that $v + \lambda w \in S$. Consequently, the set S is "shy" in X in the sense of [HSY].

Remark 2: The assumptions in Theorem 1 require information concerning both the boundary equilibria and the coexistence equilibrium. If (H5) holds and the semiflow is strongly order preserving on $C_0 \cup \{E_1, E_2\}$, the requirement that B_i contains a neighborhood of E_i in C for $i = 1, 2$ automatically follows from the other assumptions. In turn, if E_1 and E_2 are not uniform strong repellers for $[E_2, E_1]_K$, the requirement that the spectral radius of $L = D_x T_{t_0}(E)$ is larger than 1 can be replaced by the requirement that the spectral radius is different from 1 or that L is strongly positive with respect to the cone K. See Theorem 3.3 and Theorem 3.4.

Remark 3: Observe that $S = \partial B_i$ for $i = 1, 2$, where the boundary is taken relative to C. If we assume that T is **strongly monotone** on $\mathrm{Int}I = [[E_2, E_1]]_K$, that is, for some $t_0 > 0$, whenever $E_2 \ll_K x <_K y \ll_K E_1$ then $T_{t_0}x \ll_K T_{t_0}y$, and $T_{t_0}(I)$ has compact closure in X, then $S \cap [[E_2, E_1]]_K$ is a codimension one, Lipschitz manifold in the coarser order topology, normed in terms of an order unit. See Proposition 1.2 and 1.3 in [Ta1]. If T_{t_0} is a C^1 map in $[[E_2, E_1]]_K$ with strongly positive derivatives ($L(K \setminus \{0\}) \subset \mathrm{Int}K$) then it follows from a result of [Te] that $S \cap [[E_2, E_1]]_K$ is C^1 in the order topology.

We now consider the case that both E_i are unstable. We then anticipate that orbits starting in $C_0 \cap [E_2, E_1]_K$ will stay away from $C_1 \cup \{E_0\} \cup C_2$.

Theorem 2. *Let (H0)-(H3) and (H5) hold. Assume that E_1 is not locally attractive from below and E_2 is not locally attractive from above, that E_1 and E_2 are isolated equilibria, and that E_0 is an isolated compact invariant set in C. Assume that T is strongly order preserving if restricted to the positively invariant set $C_0 \cup \{E_1, E_2\}$.*

Further assume that any order bounded monotone sequence of equilibria has a limit.

Then there exist two (not necessarily different) equilibria E_1^c and E_2^c such that

$$E_2 <_K E_2^c \leq_K E_1^c <_K E_1.$$

E_1^c is the largest and E_2^c the smallest equilibrium in C_0 with respect to $\leq_K$. Let

$$D = \{x \in C_0 : \omega(x) = E_1^c \text{ or } \omega(x) = E_2^c \text{ or } E_1^c <_K \omega(x) <_K E_2^c\}.$$

Then $C_0 \subset B_1 \cup B_2 \cup D$, D is a relatively open set in C_0 containing $C_0 \cap [E_2, E_1]_K$, and the set of quasi-convergent points is dense in C_0. All orbits starting in $[E_1^c, E_1]_K \cap C_0$ converge to E_1^c, while all orbits starting in $[E_2, E_2^c]_K \cap C_0$ converge to E_2^c. Finally, if $B_j \cap C_0$ is not empty, then B_j contains a nonempty relatively open set U in C_0.

Obviously, if (H4) is assumed in Theorem 2, then $E_1^c = E_2^c \equiv E$ and $D = B$ is the basin of attraction of E.

In case the cones X_i^+ are normal, i.e. there exists $k_i > 0$ such that $0 \leq x \leq y$ in X_i implies $\|x\| \leq k_i \|y\|$ for $i = 1, 2$, then

$$\mathrm{dist}([E_2^c, E_1^c]_K, C \setminus C_0) \geq \min\{\|\hat{x}_2\|/k_2, \|\bar{x}_1\|/k_1\} > 0,$$

where $E_1^c = (\hat{x}_1, \hat{x}_2)$ and $E_2^c = (\bar{x}_1, \bar{x}_2)$. In this case, Theorem 2 implies uniform persistence (see [Th2]) for orbits starting in $C_0 \cap [E_2, E_1]_K$.

A compact invariant set $M \subset C$ is called an **isolated compact invariant set** in C, if there exists a relatively open subset U of C such that M is the only compact invariant set contained in U. The hypothesis that E_0 is an isolated compact invariant set of C holds if, for example, there exists an open set U in X containing E_0 such that for each $x \in U \cap C$ distinct from E_0, there is a $t_0 > 0$ such that $T_{t_0} x \notin U$.

We remark that the hypothesis that any order bounded monotone sequence of equilibria has a limit is satisfied if, for example, $T_t[E_2, E_1]_K$ is compact for some t.

When the E_j possess local center-stable manifolds which are graphs of functions, then we can improve the conclusion of Theorem 2.

Corollary 3. *Let the assumptions of Theorem 2 be satisfied. Further assume that C has nonempty interior and either* Int$C$ *is dense in C_0 or that $T_t(C_0) \subset$* Int$C$ *for all large t. Assume that, for $j = 1, 2$, there are $t_j > 0$ and neighborhoods U_j of E_j in X such that T_{t_j} can be extended to U_j and is continuously differentiable on U_j with the essential spectral radius of the derivative $L_j \equiv D_x T_{t_j}(E_j)$ being strictly less than 1 and its spectral radius exceeding one.*

Then $C_0 \subset D$ and any positive orbit starting at a point $x \in C_0, x \leq_K E_2^c$, converges to E_2^c, while any positive orbit starting at a point $x \in C_0, x \geq_K E_1^c$, converges to E_1^c.

Remark 4: The hypothesis that the spectral radius of L_j exceed one, $j = 1, 2$, in Corollary 3 can be weakened as described in Remark 2.

In many applications, one obtains a competitive system as a limiting system from a non-autonomous dynamical system.

A **non-autonomous semiflow** on X^+ is a continuous map $\Phi : \Delta \times X^+ \to X^+$, $\Delta = \{(t, s) : 0 \leq s \leq t < \infty\}$, satisfying

$$\Phi(t, s, \Phi(s, r, x)) = \Phi(t, r, x),\ t \geq s \geq r \geq 0,$$

and

$$\Phi(s, s, x) = x.$$

We define the omega limit set of an orbit $\{\Phi(t, s, x) : t \geq s\}$ exactly as for the autonomous case.

Following [Th1], we say that Φ is **asymptotically autonomous** with limit semiflow $T : [0, \infty) \times X^+ \to X^+$ if

$$\Phi(t_j + s_j, s_j, x_j) \to T_t(x),\ j \to \infty$$

for any three sequences $t_j \to t$, $s_j \to \infty$, and $x_j \to x$, where $x_j, x \in X^+$ and $t_j, t \geq 0$.

We consider the situation

(H6) where all ω-limit sets of the asymptotically autonomous semiflow Φ on X^+ are contained in a closed convex subset C of X^+ which is positively invariant under the limit semiflow T with (H0)-(H3) being satisfied.

We naturally ask whether the limiting behavior of its orbits mimics that for T. In general this will not be true (see [MST,Th3]), but in case that the hypotheses of Theorem 2 hold for T we establish the following.

Corollary 4. *Add to the assumptions of Theorem 2 that the basins of attraction of E_1 and E_2 do not contain open sets in C_0, e.g., add the assumptions of Corollary 3. Then every pre-compact forward orbit of an asymptotically autonomous semiflow on X^+ satisfying (H6) with limit semiflow T approaches either E_0, E_1, E_2, or $[E_2^c, E_1^c]_K$.*

Similar results are true for both the bi-stable case (Theorem 1) and in the case when competitive exclusion holds for the dynamics of T. In Theorem 4.5 we show that the omega limit set ω of an asymptotically autonomous semiflow with limit semiflow T satisfying the conclusions of Theorem 1 satisfies $\omega = \{E_1\}$ or $\omega = \{E_2\}$ or $\omega \subset S$. If there are no equilibria of T in C_0, then it follows that all orbits of T in $[E_2, E_1]_K$ converge to E_1 or all such orbits converge to E_2 (see [HSW2] or Proposition 3.6). In this case, Theorem 4.2 implies that all precompact orbits of an asymptotically autonomous semiflow with limit semiflow T converges to one of E_0, E_1, or E_2.

2. Proof of Theorem 1

In this section, we assume (H0)-(H4) hold. The following Lemma is used in the proof of Theorem 1. See Proposition 3.6 of [Sm2] for a similar version and Theorem 2.10 in [Sm3] for a finite dimensional version when E is hyperbolic.

Lemma 2.1. *Under the hypotheses of the Theorem 1, B is unordered.*

Proof: There exists a local, center-stable manifold, $W^{cs}(E)$, of E which is the graph of a Lipschitz function over the center-stable subspace of $D_xT_{t_0}(E)$. See Theorem III.8 and note exercise III.2 of [Sh]. Also, there exists a neighborhood $O \subset U$ of E such that if $T_{nt_0}x \in O$ for all integers $n \geq 0$, then $x \in W^{cs}$. Note that nt_0, n a positive integer, may replace t_0 in the hypotheses of the Theorem so we may assume that t_0 is as large as required.

Suppose that $x, y \in B$ satisfy $x <_K y$. If z satisfies $x <_K z <_K y$, then the strong order preserving property implies there exists $t_1 > 0$ and an open neighborhood V of z in C_0 such that $T_{t_1}x \leq_K T_{t_1}V \leq_K T_{t_1}y$. By a standard comparison, $V \subset B$, i.e., B has nonempty interior in C_0. Since t_1 can be chosen larger, we may assume that $t_1 = t_0$. Using the strict order preserving property, we may assume that $x, y \in O$ and $T_t[x, y]_K \subset O$ for all $t \geq 0$. Furthermore, we may assume that V is open in X and $V \subset O$ so $T_{nt_0}V \subset T_{nt_0}[x, y]_K \subset O$ for all integers $n \geq 0$. It follows that $V \subset W^{cs}(E)$. We have a contradiction since $W^{cs}(E)$ (or any graph) cannot contain an open set. ∎

Proof of Theorem 1: We observe that B_1 is open and contains all points $(x_1, 0)$ with $x_1 \neq 0$; similarly for B_2. S is a closed set containing E_0, E, and B. Furthermore, if $x_1 <_K x_2$ belong to S, then the line segment joining them belongs to S. Indeed, if $z \in L$ does not belong to S, then $z \in B_i$ for some i, say $i = 1$, implying, by comparison ($x_1 <_K z <_K x_2$), that $x_2 \in B_1$, a contradiction. S is forward invariant for T_t and T_t is strongly order preserving on S since $S \setminus \{E_0\} \subset C_0$. Finally, if $x \in S$ is a quasi-convergent point, then either $x = 0$ or x is convergent and $\omega(x) = E$.

We can apply Theorem 3.5 of [ST2] to the strongly order preserving semiflow $T_t : S \to S$, even though we do not make the hypotheses (I),(J),(M),(D),(Σ), because the stronger form of the improved Limit Set Dichotomy of [ST2], used in the proof of Theorem 3.5, automatically follows from the weaker version in [ST1] for T_t, restricted to S, because every quasiconvergent point of T_t is convergent. Hence, if J is any nontrivial linearly ordered line segment in S, then the set of points of J that are not convergent points is at most countable. By Lemma 2.1, B is unordered. Thus, J contains at most one point of B. But no point of J can converge to E_0, so J can contain at most one convergent point because the only equilibria in S are E_0 and E.

This contradiction (J is uncountable) proves that S contains no nontrivial linearly ordered line segments and hence is unordered.

If $x \in S$ is distinct from E_0 and if J is a linearly ordered line segment centered at x (x is an interior point of J), then, according to the previous paragraph, $J \setminus \{x\} \subset B_1 \cup B_2$. Thus, $B_1 \cup B_2$ is dense in C. If X is finite dimensional, then the Lebesgue measure of S is zero by the Fubini Theorem.

Finally, if $x <_K E$, then $x \in B_1 \cup B_2$ since S is unordered and $E \in S$. If $x \in B_1$ then so is E by comparison. Thus, $x \in B_2$. ∎

3. Theorem 1 revisited, Proof of Theorem 2 and Competitive Exclusion

Assumptions (H0)-(H3) are assumed to hold throughout this section. We begin with some preliminary results. Strict or strong order preserving properties of T are not needed in Proposition 3.1.

Proposition 3.1. Assume (H5) as well. Then the following hold:

(a) *For every compact set M in $C_1 \cup \{E_0\}$ there exists some $t \geq 0$ and $x \in C_1$ such that $T_t(M) \leq_K x$. For every compact set $M \subset C_2 \cup \{E_0\}$ there exists some $t \geq 0$ and $y \in C_2$ such that $y \leq_K T_t(M)$. For every compact set in $M \subset C_i$ there exists some $t \geq 0$ and $x, y \in C_i$ such that $x \leq_K T_t(M) \leq_K y$, $i = 1, 2$.*

(b) *For every compact set M in C there exist $u \in C_2, v \in C_1$ and $t \geq 0$ such that $u \leq_K T_t(M) \leq_K v$.*

(c) *$\{E_i\}$ is the only compact invariant set in C_i.*

(d) *Every compact invariant set in C is contained in $[E_2, E_1]_K$.*

Proof : (a) Using (H5) choose $x, z \in C_1$, $t \geq 0$, and a relatively open set U in C_1 such that $E_1 \in U$ and $x \leq_K T_t(U) \leq_K z$. By monotonicity, $T_s(x) \leq_K T_{t+s}(U) \leq_K T_s(z)$ for all $s \geq 0$. For every $y \in C_1$, by (H3), there exists some $t_y \geq 0$ such that $T_{t_y}(y) \in U$. By continuity, there exists a relatively open set V_y in C_1 such that $y \in V_y$ and $T_{t_y}(V_y) \in U$. So $T_s(x) \leq_K T_{t+s+t_y}(V_y) \leq_K T_s(z)$ for all $s \geq 0$. $(V_y - y) \cap (C_1 \cup \{E_0\})$ is a neighborhood of E_0 in C_1 and, by monotonicity, $T_{t+s+t_y}((V_y - y) \cap (C_1 \cup \{E_0\})) \leq_K T_s(z)$ for all $s \geq 0$. So, for every $y \in C_1 \cup \{E_0\}$, there exists a relatively open set V_y and some $s_y \geq 0$ such that $T_{s+s_y}(V_y) \leq_K T_s(z)$ for all $s \geq 0$. If M is a compact subset of $C_1 \cup \{E_0\}$, we find relative open sets $V_1, \ldots, V_k$ and $t_1, \ldots, t_k \geq 0$ such that $T_{s+t_j}(V_j) \leq_K T_s(z)$ for all $s \geq 0$, $j = 1, \ldots, k$, and $M \subset \bigcup_{j=1}^k V_j$. Since E_1 attracts all points in C_1, there exists some $s_0 \geq 0$ such that $T_s(z) \in U$ for all $s \geq s_0$, so $T_s(z) \leq_K z$ for $s \geq s_1 \equiv s_0 + t$. Hence $T_{s+t_j}(V_j) \leq_K z$ for all $s \geq s_1, j = 1, \ldots, k$. Let $\bar{s} = s_1 + t_1 + \cdots t_k$. Then $T_{\bar{s}}(V_j) \leq_K z$ for all $j = 1, \ldots, k$ and so $T_{\bar{s}}(M) \leq_K z$.

The other statements in (a) are proved similarly.

(b) Let π_i be the projections of C into $C_i \cup \{E_0\}$. Then

$$\pi_2(y) \leq_K y \leq_K \pi_1(y) \quad \forall y \in M,$$

and $\pi_i(M)$ are compact sets in $C_i \cup \{E_0\}$. By (a) there exist $s, t \geq 0$, $x \in C_2, z \in C_1$ such that

$$x \leq_K T_s(\pi_2(M)), \quad T_t(\pi_1(M)) \leq_K z.$$

By monotonicity and the inequalities above,

$$T_t(x) \leq_K T_{s+t}(M) \leq_K T_s(z)$$

and $u = T_t(x) \in C_2, v = T_s(z) \in C_1$.

(c) Let M be compact and invariant in C_i. By (a) we find $x \in C_1, z \in C_1$ and $t \geq 0$ such that $x \leq_K T_t(M) \leq_K z$. By monotonicity of T and invariance of M, $T_s(x) \leq_K M \leq_K T_s(z)$ for all $s \geq 0$. Since $T_s(x) \to E_1$ and $T_s(z) \to E_1$ as $s \to \infty$, $M = \{E_1\}$. A similar argument applies to C_2.

(d) By (b) we find $x \in C_2, z \in C_1$ and $t \geq 0$ such that $x \leq_K T_t(M) \leq_K z$. By monotonicity of T and invariance of M, $T_s(x) \leq_K M \leq_K T_s(z)$ for all $s \geq 0$. Since $T_s(x) \to E_2$ and $T_s(z) \to E_1$ as $s \to \infty$, $M \subset [E_1, E_2]_K$. ∎

Theorem 3.2. *Assume (H5) as well. Let $u_0 <_K v_0$ be two equilibria in $C_0 \cup \{E_1, E_2\}$. In case $u_0 = E_2$ and $v_0 = E_1$, assume that E_0 is an isolated compact invariant set. Assume that T is strongly order preserving if restricted to the forward invariant set $C_0 \cup \{E_1, E_2\}$.*

Then one of the following holds:

(i) *v_0 is a uniform strong repeller for $([u_0, v_0]_K \setminus \{v_0\}) \cap C_0$ and u_0 is locally attractive from above. All orbits beginning in $([u_0, v_0]_K \setminus \{v_0\}) \cap C_0$ converge to u_0.*

(ii) *u_0 is a uniform strong repeller for $([u_0, v_0]_K \setminus \{u_0\}) \cap C_0$ and v_0 is locally attractive from below. All orbits beginning in $([u_0, v_0]_K \setminus \{u_0\}) \cap C_0$ converge to v_0.*

(iii) *u_0 is locally attractive from above and v_0 is locally attractive from below.*

(iv) *There exists an equilibrium in $[u_0, v_0]_K \cap C_0$ distinct from u_0 and v_0.*

Possibility (iii) implies possibility (iv) if $[u_0, v_0]_K$ is bounded and T_t is condensing on $[u_0, v_0]_K$ for all $t > 0$.

Proof: Let J be the linearly ordered arc $ru_0 + (1-r)v_0, 0 \leq r \leq 1$, connecting u_0 and v_0. This arc lies completely in $(C_0 \cup \{E_1, E_2\}) \cap [u_0, v_0]_K =: Y$. Because T is order preserving and C_0 is positively invariant (H1), Y is positively invariant. We claim that the omega limit set of each of its points is contained in Y. To see this, first observe that any omega limit point of a point of Y must belong to $[u_0, v_0]_K$ because the latter is closed and by the monotonicity of T. If either u_0 or v_0 belongs to C_0, then $E_0 \notin [u_0, v_0]_K$, so E_0 cannot be an omega limit point of any point of Y. If $u_0 = E_2$ and $v_0 = E_1$, then we argue that E_0 cannot belong to an omega limit set of a point of Y using the Butler McGehee lemma (Proposition 4.1 in [Th2]) and the hypothesis that $\{E_0\}$ is an isolated compact invariant set. Indeed, $\omega(x) \neq \{E_0\}$ by (H2), so if $E_0 \in \omega(x)$, then there must exist $y \in \omega(x)$ distinct from E_0 such that $\omega(y) = \{E_0\}$, a contradiction to (H2). Therefore, we conclude that E_0 cannot belong to the omega limit set of a point of Y. If an omega limit set ω of a point of Y were to contain a point belonging to C_1, then $\omega \cap C_1$ would be a compact and invariant subset of C_1. By Proposition 3.1, $\omega \cap C_1 = \{v_0\}$. Similar reason shows that ω can contain no point of C_2 other than E_2. It follows that the omega limit set of a point of Y is a subset of Y. Further T is strongly order preserving on Y. Let us exclude possibility (iv). Applying Theorem 3.5 in [ST2] to T_t, restricted to Y (which contains at most two equilibria), all but countably many points in J are convergent points. By a comparison argument, there can be at most one nonconvergent point so we have the following three cases:

(i) All orbits starting on $J \setminus \{v_0\}$ converge to u_0. Let $u_0 <_K x <_K v_0, x \in C_0$. Then there exist relatively open sets U, V in $C_0 \cup \{E_1, E_2\}$ and $t_0 > 0$ such that $u_0 \in U$, $v_0 \in V$ and $T_{t_0}(U) \leq_K T_{t_0}x \leq_K T_{t_0}(V)$. Since V is relatively open, there exists some $w \in J \cap V, w <_K v_0$. By comparison with $T(t)w \to u_0$, $T(t)(U) \to u_0, T(t)x \to u_0$ as $t \to \infty$. Thus, u_0 is locally attractive from above. Recall that, if $u_0 = E_2$, we already know that it is globally attracting for C_2. Obviously v_0 is a uniform strong repeller for $C_0 \cap [u_0, v_0]_K$.

(ii) All orbits starting on $J \setminus \{u_0\}$ converge to v_0. The proof is analogous to (i).

(iii) There exists w in $J \setminus \{u_0, v_0\}$ such that $\omega(x) = \{u_0\}$ for all $x \in J, x < w$ and $\omega(x) = \{v_0\}$ for all $x \in J, x >_K w$. Since T is strongly order preserving on $C_0 \cup \{E_1, E_2\}$, u_0 is locally attractive from above and v_0 is locally attractive from below, in $C_0 \cup \{E_1, E_2\}$ by an argument similar to case(i). Recalling that the boundary equilibria are attracting in the part of the 'boundary' they lie in, u_0 is locally attractive from above and v_0 is locally attractive from below, in $[u_0, v_0]_K$.

If T_t is condensing for all $t > 0$ and $[u_0, v_0]_K \neq [E_2, E_1]_K$ is bounded, the same proof as in Proposition 3.7 in [ST1] shows that there exists an equilibrium in $[u_0, v_0]$

distinct from u_0 and v_0. The assumption of boundedness of $[u_0, v_0]_K$ allows us to drop the assumption of normality in the above mentioned proof. If $[u_0, v_0]_K = [E_2, E_1]_K = I$ is bounded and T_t is condensing on I for each $t > 0$, then the proof of Proposition 3.7 in [ST1] can be modified to show that there exists an equilibrium other than E_0, E_1, E_2 in I. The first modification in that proof is to require that E_0 (as well as $u_0 \equiv E_2$ and $v_0 \equiv E_1$) belong to each closed convex set K in the family χ. (We use notation of the proof of Proposition 3.7 in [ST1] here; in particular, K represents closed convex subsets of I in that proof and not the cone $X_1 \times (-X_2)$.) We will show below that E_0 is a uniform strong repeller for $I \setminus \{E_0\}$, so there exists $\delta > 0$ such that $\liminf_{t\to\infty} \|T_t(x) - E_0\| > \delta$ for all $x \in I \setminus \{E_0\}$. Therefore, E_0 is an ejective fixed point of T_{t_0} on K in the sense that if $B(E_0) = \{x \in K : \|x - E_0\| < \delta/2\}$, then for each $x \in B(E_0)$ distinct from E_0, there exists a positive integer n such that $T_{nt_0}(x) \notin B(E_0)$. We may choose δ so small that $B(E_0)$ has empty intersection with $B(u_0)$ and $B(v_0)$. Furthermore, as E_0 is an extreme point of I, it is also an extreme point of the convex set $K \subset I$. Therefore, by a result of Nussbaum [Nu2], the fixed point index $i(T_{t_0}, B(E_0)) = 0$. The argument in [ST1] using the additivity of the fixed point index now implies the existence of a fixed point of T_{t_0} belonging to $K \setminus (B(u_0) \cup B(v_0) \cup B(E_0))$. Clearly, this fixed point is distinct from E_0, E_1, E_2. Now the argument in [ST1] continues unchanged, keeping in mind that the radii of the balls $B(u_0), B(v_0), B(E_0)$ are independent of $t_0 > 0$, establishing the existence of an equilibrium for T in $[E_2, E_1]_K$ distinct from E_0, E_1, E_2.

It remains only to show that E_0 is a uniform strong repeller for $I \setminus \{E_0\}$. Let $r > 0$ and let A_0 be the omega-limit set of I under the condensing map T_r. If follows from [Ha], Lemma 2.3.5 and Corollary 2.2.4, that A_0 is non-empty, compact, invariant, and attracts I under T_r. In particular $T_{k_j r} x_j \to A_0$ as $j \to \infty$, `for all sequences` (x_j) `in` I `and` $(k_j) \in \mathbf{N}$ `such that` $k_j \to \infty$ `as` $j \to \infty$. Let (t_j) be a sequence in $[0, \infty)$, $t_j \to \infty$ as $j \to \infty$, and (y_j) a sequence in I. Then $t_j = k_j r + s_j$ with $k_j \in \mathbf{N}$,

$k_j \to \infty$, and $s_j \in [0, r)$. Now

$$T_{t_j} y_j = T_{k_j r}(T_{s_j} y_j) \to A_0$$

because $x_j = T_{s_j} y_j \in I$. In the following we use the terminology of [Th2], in particular of Section 4.2. We have just shown that condition $C_{4.2}$ in [Th2] is satisfied for I. Set $I_2 = \{E_0\}$ and $I_1 = I \setminus I_2$. Since I_2 is an isolated compact invariant set of I by assumption, it is an isolated covering of itself. Further, by (H2), I_2 is a weak repeller for I_1. Theorem 4.6 in [Th2] implies that I_2 is a uniform strong repeller for I_1. ∎

By assuming the strong order preserving property on $C_0 \cup \{E_1, E_2\}$ rather than only on C_0, we may drop the assumption that B_i contains a neighborhood of E_i, $i = 1, 2$, in Theorem 1 because it is a consequence of the stronger assumption.

Theorem 3.3. *Let (H0)-(H5) hold and T be strongly order preserving on $C_0 \cup \{E_1, E_2\}$. Suppose that E is an interior point of C_0 (in particular X^+ and K have nonempty interior) and there is a neighborhood U of E in X with $U \subset C_0$ such that T_{t_0} is a C^1 map in U and the radius of the essential spectrum of $D_x T_{t_0}(E)$ is strictly less than one for some positive t_0. Then the assertions (a), (b), and (c) in Theorem 1 hold, if at least one of the following two assumptions is satisfied:*

(i) The spectral radius of $D_x T_{t_0}(E)$ is strictly greater than one.

(ii) E is not locally attractive from above or from below, and the spectral radius of $D_x T_{t_0}(E)$ is different from one.

Sometimes it is easier to check the stability of the boundary equilibria than of the interior equilibrium. So we mention that E is not locally attractive from above or below (see (ii)) under the assumptions of this theorem, if E_1 and E_2 are not uniform strong repellers for $[E_2, E_1]_K$ and T_t is condensing for all $t > 0$. (See Theorem 3.2.)

Proof: Applying Theorem 3.2 to $u_0 = E_2, v_0 = E$ and $u_0 = E, v_0 = E_1$, in case (ii) above implies that all orbits starting in $[E_2, E]_K \setminus \{E\}$ converge to E_2, while all orbits starting in $[E, E_1]_K \setminus \{E\}$ converge to E_1. This implies that the spectral radius of $D_x T_{t_0}(E)$ is greater than 1 also under assumption (ii). All assumptions of Theorem 1 are satisfied, once we show that B_i contains a neighborhood of E_i for $i = 1, 2$.

Since T is strongly order preserving on $\tilde{C} = C_0 \cup \{E_1, E_2\}$, there exists an open subset $U \ni E_1$ and some $r > 0$ such that $T_r(U \cap \tilde{C}) \geq_K T_r(x)$, $x = (1/2)(E_1 + E)$. By monotonicity, the inequality holds for all $t \geq r$. Since $T_t x \to E_1$ as $t \to \infty$, we have $\omega(y) \geq_K E_1$ for all $y \in U \cap \tilde{C}$. But $\omega(y) \subset [E_2, E_1]_K$ for all $y \in C$ and so $\omega(y) = \{E_1\}$ for all $y \in U \cap \tilde{C}$. If the open set U is chosen small enough to be disjoint from $\{E_0\} \cup C_2$, we also have $\omega(y) = \{E_1\}$ for all $y \in U \cap C_1$ by (H3) and thus for all $y \in U \cap C$. In other words, $U \cap C \subset B_1$. Similarly we show that B_2 contains a neighborhood of E_2. ∎

Remark. In many applications additional information is available to rule out that the spectral radius of $D_x T_{t_0}(E)$ equals one. So let us assume that the spectral radius of $D_x T_{t_0}(E)$ equals one. A Krein-Rutman type result by Nussbaum [Nu1] implies that 1 is an eigenvalue of $D_x T_{t_0}(E)$ with an eigenvector in $K \setminus \{E_0\}$. Let us assume in addition that T_t is differentiable at E for all $t \geq 0$. Then the linear operators $T'(t) = D_x T_t(E)$ form a semigroup on X under which the cone K is forward invariant. It does not follow from our assumptions, but will turn out to be the case in many applications that $T'(t)$ is a C_0-semigroup; so let us assume that this is the case and let A denote the infinitesimal generator of T'. By the spectral mapping theorem for C_0-semigroups, 0 is an eigenvalue of A with an eigenvector v in $K \setminus \{E_0\}$. In many applications, already this can be shown to be incompatible with E being a fixed point of T, or to be not generic.

Under additional assumptions, the case where the spectral radius equals one is treated in our next result.

Theorem 3.4. *Assume (H0)-(H5) and T to be strongly order preserving on $C_0 \cup \{E_1, E_2\}$. Let the unique coexistence equilibrium E be an interior point of C_0 (i.e., X^+ and K have nonempty interior), and assume that there is a neighborhood U of E in X with $U \subset C_0$ such that T_{t_0} is a C^1 map in U and the radius of the essential spectrum of the derivative $L = D_x T_{t_0}(E)$ is strictly less than one for some positive t_0. Assume that the K-positive operator L is strongly K-positive, i.e., for every $u \in K \setminus \{E_0\}$ there exists some $n \in \mathbf{N}$ such that $L^n u \in \text{Int } K$.*

Finally assume that E is not locally attractive from above or from below.

Then the assertions (a), (b), and (c) in Theorem 1 hold.

Again we mention that the assumption of E being not locally attractive from above or from below follows under the other assumptions of this theorem, if E_1 and E_2 are not uniform strong repellers for $[E_2, E_1]_K$ and T_t is condensing for every $t > 0$.

Proof: From the proof of Theorem 3.3 we learn that the spectral radius of L is greater than or equal to one. If it is strictly greater than one, the assertion follows from Theorem 3.3. So we assume that the spectral radius of L is one. From the preceding remark we learn that 1 is an eigenvalue of L with an eigenvector v in K. It follows from Theorem 2.10 and Theorem 2.13 in [Kr] that 1 is a simple eigenvalue of L with one-dimensional eigenspace and that all other eigenvalues of L have their absolute values strictly less than 1. Since the radius of the essential spectral of L is strictly less than one, there exists some $p \in (0, 1)$ such that all spectral values of L different from one have their absolute values in $[0, p)$. Further the eigenvector $v \in K$ lies in Int K. The above implies that there exists a projection P which commutes with L and an eigenvector $v^* \in K^*$ of L^* such that

$$\langle v^*, v \rangle = 1, \qquad x = \langle v^*, x \rangle v + Px \quad \forall x \in X,$$

and

$$\|L^n P x\| \le M p^n \|x\| \quad \forall x \in X$$

with some appropriate number $M > 0$. Further, since $v \in \text{Int } K$, there exists some $c > 0$ such that $-c\|x\|v \le_K x \le_K c\|x\|v$ for $x \in X$.

It follows from Theorem 3.2 that all orbits starting in $[E_2, E]_K \setminus \{E\}$ converge to E_2, while all orbits starting in $[E, E_1]_K \setminus \{E\}$ converge to E_1. Using the strong order preserving property, $E_1 \le_K \omega(x)$ for every $E <_K x$ and $\omega(x) \le_K E_2$ for every $x <_K E$. Since $\omega(x) \in [E_2, E_1]_K$ for all $x \in C$, we have that all orbits starting at some $x <_K E$ converge to E_2, while all orbits starting at some $x >_K E$ converge to E_1.

Now let $x \in B$. By the chain rule, T_{nt_0} is differentiable at E with derivative L^n. So there exists some function ψ_n defined in a neighborhood W_n of E_0 such that

$$\begin{aligned} T_{nt_0}x - E =& L^n(x - E) + \|x - E\| \, \psi_n(x - E) \qquad \forall x \in E + W_n, \\ &\psi_n(y) \to 0, y \to 0. \end{aligned}$$

Then, using $Lv = v$,

$$\begin{aligned} T_{nt_0}x - E =& \langle v^*, x - E\rangle v + L^n P(x - E) + \|x - E\| \, \psi_n(x - E) \\ \ge_K& \langle v^*, x - E\rangle v - c\|L^n P(x - E)\| \, v - c\|x - E\| \, \|\psi_n(x - E)\| \, v \\ \ge_K& \Big(\langle v^*, x - E\rangle - cMp^n\|x - E\| - c\|x - E\| \, \|\psi_n(x - E)\|\Big) v. \end{aligned}$$

A similar inequality may be obtained with $\ge_K$ replaced by $\le_K$ and $-$ replaced by $+$ in front of the two terms with factor c. We conclude that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|\langle v^*, x - E\rangle| \le \epsilon\|x - E\| \quad \text{whenever } x \in B, 0 < \|x - E\| < \delta.$$

Otherwise, by choosing first n large and then $\delta > 0$ small, and assuming that $\langle v^*, x - E\rangle > \epsilon\|x - E\|$, we can achieve that $0 <_K T_{nt_0}x - E$ for some $x \in B$, which means that $T_t x \to E_1$ as $t \to \infty$, contradicting $x \in B$. Similarly we deal with the case that $\langle v^*, x - E \rangle < -\epsilon \|x - E\|$. Now

$$\|T_{nt_0} x - E\| \leq |\langle v^*, x - E \rangle| \|v\| + \|L^n P(x - E)\| + \|x - E\| \|\psi_n(x - E)\|.$$

Hence, for $\epsilon > 0$ we find $\delta > 0$ such that for $x \in B$ and $0 < \|x - E\| < \delta$, we have $x \in E + W_n$ and

$$\|T_{nt_0} x - E\| \leq \epsilon \|v\| \|x - E\| + M p^n \|x - E\| + \|x - E\| \|\psi_n(x - E)\|.$$

Choosing $\epsilon > 0$ small enough and n large enough and finally $\delta > 0$ small enough,

$$\|T_{nt_0} x - E\| \leq \frac{1}{2} \|x - E\|, \qquad \text{whenever } \|x - E\| < \delta, \quad x \in B.$$

This means that, for $x \in B$ and large t, $T_t x$ is in the local strongly stable manifold of the map T_{t_0} at E which is a Lipschitz graph and so does not contain open sets [Sh, Theorem III.8 and Exercise III.2]. A similar argument as in Lemma 2.1 shows that B is unordered and the proof now proceeds as the proof of Theorem 1. ∎

Remark. If $T_t x$ is differentiable in x at E for all $t \geq 0$ and the derivatives $T'(t)$ define a C_0-semigroup with infinitesimal generator A, there may be alternative assumptions that are easier to check than the strong positivity of L. Since the radius of the essential spectrum of $D_x T_{t_0}(E)$ is strictly smaller than one, T' is essentially norm-continuous [Th4] and the eigenvalue 0 of A has finite algebraic multiplicity. Recall that under the assumptions of Theorem 3.4, the open interior of K is not empty. Let us assume that for every $u \in K$ there exists some $\lambda > 0$ such that $(\lambda - A)^{-1} u \in \text{Int } K$. It follows that 0 is a first order pole of the resolvents of A with a one-dimensional eigenspace that is spanned by an eigenvector $v \in \text{Int } K$. Theorem 3.4 in [Th4] implies that there are $M, \delta > 0$ and an eigenvector $v^* \in K^*$ of A^* associated with 0 such that

$$\langle v^*, v \rangle = 1, \qquad \|T'(t)x - \langle v^*, x \rangle v\| \leq M \|x\| e^{-\delta t} \qquad \forall t \geq 0, x \in X.$$

This means we are in the same situation as in the proof of Theorem 3.4.

Alternatively let us assume that the ordered Banach spaces X_1 and X_2 are Banach lattices. Then X with cone K is a Banach lattice as well, where $|(x_1, x_2)|_K = (|x_1|, -|x_2|)$. If T' is an irreducible semigroup (or equivalently the resolvents of A are irreducible), 0 is a first order pole of the resolvent of A with a one-dimensional eigenspace which is spanned by an eigenvector $v \in \text{Int } K$. See de Pagter [Pa1], Proposition 7.6, Heijmans [Hei], Theorem 8.17, and de Pagter [Pa2], Proposition A.2.10. Again we have the asymptotic behavior of T' that makes the proof of Theorem 3.4 work.

Proof of Theorem 2: A similar argument to that given in the proof of Theorem 3.2 establishes that for each $x \in C_0 \cup \{E_1, E_2\}$, $\omega(x) \subset C_0 \cup \{E_1, E_2\}$.

Let J be the ordered arc $x_r = (1-r)E_2 + rE_1, \quad 0 \leq r \leq 1$. The same proof as in Theorem 3.5 in [ST2] shows that all but countably many points in J are quasiconvergent points. Since E_1 is not locally attractive from below, $\omega(x_r) <_K E_1$ for all $r \in [0, 1)$. In fact, $\omega(x_r) \leq_K E_1$ by the limit set dichotomy of [ST1] and if $E_1 \in \omega(x_r) \leq_K E_1$, then $\omega(x_r) = \{E_1\}$ by the nonordering of limit sets [ST1]. But if $\omega(x_r) = \{E_1\}$, then the strong order preserving property implies that E_1 is locally attracting from below, a contradiction. Hence, $\omega(x_r) <_K E_1$. Since E_1 is an isolated equilibrium and because $0 \leq r < s \leq 1$ implies that $\omega(x_r) \leq_K \omega(x_s)$ by the limit set dichotomy, $\bigcup_{r \in [0,1)} \omega(x_r)$ is bounded away from E_1. For if not, there exists a sequence $\{x_n\}$ with $x_n \in \omega(x_{r_n})$ for $r_n \in [0, 1)$. As all but countably many points of J are quasiconvergent, we may use the limit set dichotomy to choose $1 > r'_n > r_n$ so that $r'_{n+1} > r'_n$ and $y_n \in \omega(x_{r'_n})$ such that y_n is an equilibrium. Then $x_n <_K y_n$ and $E_2 \leq_K y_n <_K y_{n+1} <_K E_1$. By our hypothesis, $\{y_n\}$ must converge and the inequality above implies that $y_n \to E_1$, contradicting that E_1 is an isolated equilibrium.

By the limit set dichotomy there are two cases:

Case 1: There exists some $r_1 \in (0,1)$ such that $\omega(x_r) = \omega(x_{r_1})$ for all $r \in [r_1, 1)$ and $\omega(x_{r_1})$ consists of equilibria. We set $\omega_1 = \omega(x_{r_1})$ in this case.

In fact, ω_1 consists of a single equilibrium. If $E \in \omega_1$, then $E <_K E_1$ and the strong order preserving property implies the existence of a neighborhood U of E_1 in $C_0 \cup \{E_1, E_2\}$ and $t_0 \geq 0$ such that $E = T_t E \leq_K T_t(U)$ for $t \geq t_0$. As $x_r \in U$ for r near 1, we conclude from the limit set dichotomy that $E \leq_K \omega_1 = \omega(x_{r_1})$. By the nonordering of limit sets (see [ST1]), $\omega_1 = \{E\}$.

Case 2: For every $r \in (0,1)$ there exists some $s \in (r, 1)$ such that $\omega(x_r) <_K \omega(x_s)$.

Because all but countably many of the points x_r are quasi-convergent, we can find a sequence $1 > r_j \nearrow 1$, $j \to \infty$, such that $\omega(x_{r_j})$ consists of equilibria and $\omega(x_{r_i}) <_K \omega(x_{r_j})$ for $j > i$. Pick equilibria $\tilde{E}_j$ from $\omega(x_{r_j})$, these form a strictly increasing sequence that is order bounded by E_2 and E_1 and bounded away from E_1 and E_2. We let ω_1 be the singleton consisting of the limit of this sequence which exists by our hypothesis. It is easy to see that ω_1 does not depend on the choice of the sequences r_j and $\tilde{E}_j$, and $\omega_1 <_K E_1$.

In cases 1 and 2, we define E_1^c by $\omega_1 = \{E_1^c\}$.

(a) We show that all orbits starting in $[E_1^c, E_1]_K \cap C_0$ converge to E_1^c.

Let $x \in C_0$ satisfy $x \in [E_1^c, E_1]_K$. Then there exists an open set W in $C_0 \cup \{E_1, E_2\}$ and some $t_1 \geq 0$ such that $E_1 \in W$ and $E_1^c \leq_K T_t(x) \leq_K T_t(W)$ for all $t \geq t_1$. Now $x_r \in W$ for $r \in (0,1)$ that are close enough to 1, and $E_1^c \leq_K T_t(x) \leq_K T_t(x_r)$ for all $t \geq t_1$ for $r \in (0,1)$ close enough to 1. By the limit set dichotomy, $E_1^c \leq_K \omega(x) \leq_K \omega(x_r)$ holds for each $r \in (0,1)$ being close enough to 1. Both in case 1 and case 2 we have $E_1^c \leq_K \omega(x) \leq_K \omega_1 = \{E_1^c\}$. Obviously, $\omega(x) = \{E_1^c\}$.

Similarly we find $\omega_2 \subset [E_2, E_1]_K$, $E_2 < \omega_2 < E_1$, and consisting of a single equilibrium denoted by E_2^c, such that all orbits starting in $[E_2, E_2^c]_K$ converge to E_2^c.

(b) We show that $D \supseteq C_0 \cap [E_2, E_1]_K$.

Let $x \in C_0$, $E_2 <_K x <_K E_1$. Then there exist open set W in C and some $t_1 \geq 0$ such that $E_1 \in W$, and $T_t(x) \leq_K T_t(W) \quad \forall t \geq t_1$. Again $x_r \in W$ for $r \in (0,1)$ close enough to 1. By the limit set dichotomy, $\omega(x) \leq_K \omega(x_r)$ for $r \in (0,1)$ close enough to 1. Both in case 1 and case 2, we have that $\omega(x) \leq_K \omega_1 = \{E_1^c\}$.

Since an analogous statement holds for ω_2, we have that $\omega(x) \subset [E_2^c, E_1^c]_K$ for all $x \in [E_2, E_1]_K \cap C_0$. In other words: $D \supseteq C_0 \cap [E_2, E_1]_K$.

(c) We show that D is relatively open in C_0.

Let $x \in D$. Then $E_2 <_K \omega(x) <_K E_1$ and $\omega(x) \subset C_0$. Since T is strongly order preserving on $C_0 \cup \{E_1, E_2\}$, we find $t_1 \geq 0$ and relatively open sets U, V, W in $C_0 \cup \{E_1, E_2\}$ such that $E_2 \in U, \omega(x) \subset V, E_1 \in W$ and $T_t(U) \leq_K T_t(V) \leq_K T_t(W)$ for all $t \geq t_1$. Choose $u \in U, w \in W$, $u, w \in C_0 \cap [E_2, E_1]_K$. Then $u, w \in D$ and, by the limit set dichotomy, for every $v \in V$, $\omega(v) = \omega(u)$ or $\omega(v) = \omega(w)$ or $\omega(u) <_K \omega(v) <_K \omega(w)$, so $v \in D$. Thus $V \subset D$. Since V is relatively open and contains $\omega(x)$, $T_t(x) \in V$ for some $t > 0$. Let $\tilde{V} = T_t^{-1}(V)$. Then $\tilde{V}$ is a relatively open neighborhood of x and $\tilde{V} \subset D$.

(d) We show that $C_0 \subset B_1 \cup B_2 \cup D$.

Let $x \in C_0$. Then $\omega(x) \subset [E_2, E_1]_K$. Suppose that $\omega(x) \cap C_0 \neq \emptyset$. Then, by (b), $\omega(x) \cap D \neq \emptyset$. Since D is relatively open, $T_t(x) \in D$ for some $t > 0$. Since $\omega(x) = \omega(T_t(x))$, $x \in D$ by the definition of D.

So we can assume that $\omega(x) \subset C_1 \cup C_2 \cup \{E_0\}$. But $\omega(x) \subset C_0 \cup \{E_1, E_2\}$ by the first paragraph of the proof so by the connectedness of $\omega(x)$ we conclude that $x \in B_1 \cup B_2$.

It follows from (d) that E_1 is the largest and E_2 the smallest equilibrium in C_0 with respect to $\leq_K$.

Finally assume that E_1 attracts a point $x = (x_1, x_2) \in C_0$. Then $(x_1, x_2) <_K (x_1, (1/2)x_2) <_K (x_1, 0)$. Since T is strongly order preserving on $C_0 \cup \{E_1, E_2\}$, there exist an open neighborhood U of $(x_1, (1/2)x_2)$ in C_0 and some $t_0 > 0$ such that $T_t x \leq_K T_t(U)$ for all $t \geq t_0$. By comparison, for every $u \in U$, $E_1 \leq_K \omega(u)$, i.e., $\omega(u) \subset C_1$. By Proposition 3.1, $\omega(u) = \{E_1\}$. Thus, $U \subset B_1$. The same proof as in Theorem 3.5 of [ST2] implies that the quasiconvergent points are dense in C_0. ∎

Proof of Corollary 3: We argue as in the proof of Lemma 2.1. Under our assumptions the local center-stable manifold of the fixed point E_j of T_{t_j} is a graph (see Theorem III.8 and note exercise III.2 of [Sh]) and so cannot contain open sets. If B_1 contains a point of C_0, then by the final paragraph of the proof of Theorem 2 it must contain a relatively open set U in C_0. In fact, returning to the argument in that last paragraph and using strict monotonicity, we may assume that U is contained in an arbitrarily small neighborhood of E_1 by replacing x by $T_{nt_1}x$ for large enough n. If Int$C$ is dense in C_0, then U contains a point of Int$C$ so we may replace U by an X-open set $V \subset U$. If $T_t C_0 \subset \text{Int}C$ for all large t then we may assume that $x \in \text{Int}C$ and so U may be taken to be open in X. In either case, we have that the local center-stable manifold of E_1 for the map T_{t_0} must contain an open subset of X, a contradiction. Therefore, B_1 contains no point of C_0. Similarly for B_2. By Theorem 2, we conclude that $C_0 \subset D$.

Let $x \in C_0$, $x \leq_K E_2^c$. Then $\omega(x) \subset [E_2^c, E_1^c]_K$ and, by monotonicity, $\omega(x) \leq_K E_2^c$, so $\omega(x)$ is the singleton consisting of E_2^c. The convergence statement concerning E_1^c follows analogously. ∎

If (H4) holds, Theorem 3.2 restricts the possible dynamics on the order intervals $[E_2, E]_K$ and $[E, E_1]_K$. If we assume that for each interval, at most one of the equilibria is locally attracting from the appropriate direction, then there are four cases to consider. Theorem 1 and Theorems 3.3 and 3.4 address the case that E is a uniform strong repeller for both intervals. Theorem 2 treats the case that E is locally attractive for both. If E were hyperbolic, these would constitute the only generic cases. However, in the nonhyperbolic case it may occur that E is a uniform strong repeller on one interval and locally attracting on the other. The result below addresses these two cases.

Theorem 3.5. *Let (H0)-(H5) hold and suppose that T is strongly order preserving if restricted to the forward invariant set $C_0 \cup \{E_1, E_2\}$. Assume that E_2 attracts all orbits in $[E_2, E]_K \cap C_0 \setminus \{E\}$ and that E attracts all orbits in $[E, E_1]_K \cap C_0$ (see Theorem 3.2 for sufficient conditions). Let*

$$S = C_0 \setminus (B_2 \cup B).$$

Then $S \cap [E_2, E_1]_K$ is an unordered, positively invariant set containing no convergent points and $B_2 \cup B$ is dense in $[E_2, E_1]_K \cap C_0$. Furthermore, $\omega(x) \subset [E_2, E]_K$ for each $x \in C_0 \cap [E_2, E_1]_K$ and $\{x \in C : x <_K E\} \subset B_2$. An analogous result holds if E_1 attracts all orbits in $[E, E_1]_K \cap C_0 \setminus \{E\}$ and E attracts all orbits in $[E_2, E]_K \cap C_0$.

Note that S need not be closed, because B need not be open.

Proof: The positive invariance of S is obvious. By the strong order preserving property, if $x \in C_0 \cap [E_2, E_1]_K$, there exists a neighborhood U of E_1 in X and $t_0 \geq 0$ such that $T_t x \leq_K T_t(U \cap C_0)$ for $t \geq t_0$. Since $U \cap C_0$ contains a point y of $[E, E_1]_K$ distinct from E_1 and since $T_t y \to E$ as $t \to \infty$, we conclude that $\omega(x) \leq_K E$ for all $x \in C_0 \cap [E_2, E_1]_K$. If $Y = (C_0 \cap [E_2, E_1]_K) \cup \{E_2, E_1\}$, then Y is positively invariant, T is strongly order preserving on Y, and, by the conclusion above and by Proposition 3.1, Y contains the omega limit set of each of its points. (Cf. the beginning of the proof of Theorem 3.2.)

If $x \in C$ and $x <_K E$, then using the strong order preserving property and the fact that E_2 attracts points in $[E_2, E]_K \setminus \{E\}$ we conclude that $\omega(x) \leq_K E_2$ and so $\omega(x) \subset C_2$. By Proposition 3.1 (c), $T_t x \to E_2$.

Observe that $S \cap [E_2, E_1]_K$ contains no convergent points since any such points must converge to E_1 but, as shown above, B_1 contains no point of $C_0 \cap [E_2, E_1]_K$, hence no point of S. If $x, y \in S \cap [E_2, E_1]_K$ and $x <_K y$, consider the line segment $J = \{sx + (1-s)y : s \in [0,1]\} \subset Y$. At most countably many points of J are nonconvergent points by Theorem 3.5 [ST2] applied to T on Y. No point of J can converge to E_2 since then, by comparison, $x \in B_2$, a contradiction. No point of J can converge to E_1 or to E_0 since $J \subset C_0 \cap [E_2, E_1]_K$. Therefore, all but countable many points of J are convergent to E. By comparison again, we conclude that all points of J except x and y converge to E. Put $z = (x+y)/2$. Then $E_2 <_K z <_K y <_K E_1$ and by a standard argument using the strong order preserving property, y must be convergent to E since points arbitrarily near E_1 in $[E_2, E_1]_K \cap C_0$ have this property. This contradiction establishes that $S \cap [E_2, E_1]_K$ is unordered.

If $x = (x_1, x_2) \in S \cap [E_2, E_1]_K$ then $x_i \neq 0$ so if $0 < s < 1$ and $y = (sx_1, x_2)$, then $y \in C_0 \cap [E_2, E_1]_K$ and $x <_K y$. Since $S \cap [E_2, E_1]_K$ is unordered, y must belong to $B_2 \cup B$. As s may be taken arbitrarily close to one, we conclude that $B_2 \cup B$ is dense in $[E_2, E_1]_K \cap C_0$. ∎

Remark: More can be said if we assume that $B_1 \cap C_0 = \emptyset$. The latter holds if we assume smoothness of T near E_1 together with spectral assumptions on the derivative as in Corollary 3. In this case, our assertions regarding $S \cap [E_2, E_1]_K$ hold for S and, in addition, we may conclude that $\{x \in C_0 : E \leq_K x\} \subset B$.

Now consider the case where T has no equilibria in C_0. Compare the following with Theorem B of [HSW2] where it is assumed that $\text{Int} X_i^+ \neq \emptyset$ for $i = 1, 2$.

Proposition 3.6. *Assume (H0)-(H5), that T is strongly order preserving if restricted to the forward invariant set $C_0 \cup \{E_1, E_2\}$, and that $\{E_0\}$ is an isolated compact invariant set.*

Suppose that there are no equilibria of T in C_0 and let one of the following two assumptions hold in addition:

(i) *$[E_2, E_1]_K$ is bounded and T_t is condensing for each $t > 0$.*

(ii) *E_2 is not locally attractive from above or E_1 is not locally attractive from below.*

Then either all orbits starting in $[E_2, E_1]_K \cap C_0$ converge to E_1 or all such orbits converge to E_2. Furthermore, $C_0 \subset B_1 \cup B_2$ and, if $B_i \cap C_0$ is nonempty, then B_i contains a nonempty open set in C_0, for $i = 1, 2$.

Proof: Theorem 3.2 implies the first assertion. Suppose for definiteness that all orbits of points of $C_0 \cap [E_2, E_1]_K$ converge to E_1. If $x \in C_0 \setminus [E_2, E_1]_K$ then $\omega(x) \subset [E_2, E_1]_K$. $E_0 \notin \omega(x)$ since $\omega(x) \neq \{E_0\}$ by (H2) and if the assertion were false, the Butler McGehee lemma would imply the existence of a $u \in \omega(x)$, $u \neq E_0$, such that $\omega(u) = E_0$, a contradiction to (H2). If $u \in \omega(x) \cap C_0$, then $E_2 <_K u <_K E_1$ so there is a neighborhood U of u, not containing any points of $C_2 \cup \{E_0\}$, and $t_0 \geq 0$ such that $E_2 \leq_K T_t(U) \leq_K E_1$ for $t \geq t_0$. Clearly, $\omega(y) = E_1$ for all $y \in U$ by monotonicity and the first assertion of the proposition. But $T_t x \in U$ for some $t > 0$ so $\omega(x) = E_1$. Consequently, we may assume that $\omega(x) \cap C_0 = \emptyset$. By connectedness of $\omega(x)$, $\omega(x) \subset C_i$ for some i. By Proposition 3.1, $\omega(x) = \{E_i\}$.

The assertion that B_i contains a nonempty open set in C_0 if $B_i \cap C_0$ is nonempty follows in the same way as in Theorem 2. ∎

We remark that in the case that $C_0 \cap [E_2, E_1]_K \subset B_1$ in Proposition 3.6, $B_2 \cap C_0$ may be nonempty, in which case it contains a nonempty open subset of C_0. A hypothesis like that of Corollary 3 can be used to conclude that $B_2 \cap C_0 = \emptyset$. Similar remarks apply to the case that $C_0 \cap [E_2, E_1]_K \subset B_1$.

For completeness we mention the following case which cannot occur if T has the compactness properties (i) in Proposition 3.6. The Proof is similar to the one in Proposition 3.6.

Proposition 3.7. Assume (H0)-(H5), that T is strongly order preserving if restricted to the forward invariant set $C_0 \cup \{E_1, E_2\}$, and that $\{E_0\}$ is an isolated compact invariant set. Suppose that both E_1 and E_2 are locally attractive from below or above respectively and that there are no equilibria on C_0. Then the union $B_1 \cup B_2$ of the basins of attraction of E_1 and E_2 is open and dense and $S = C_0 \setminus (B_1 \cup B_2)$ is unordered.

4. Asymptotically Autonomous Competitive Systems

The focus of this section is on asymptotically autonomous competitive systems with limit semiflow T, satisfying (H0)-(H3) and (H5). We require some preliminary results concerning the dynamics of T.

Proposition 4.1. *Let the assumptions of Theorem 2 be satisfied. Then any compact invariant subset M in $C_0 \cap [E_2, E_1]_K$ is contained in $[E_2^c, E_1^c]_K$.*

Proof: Let M be a compact invariant subset of $C_0 \cap [E_2, E_1]_K$. In particular $E_2 <_K M <_K E_1$. Since T is strongly order preserving on $C_0 \cup \{E_1, E_2\}$, there exist relatively open subsets U, V, W of $C_0 \cup \{E_1, E_2\}$ and some $t_1 \geq 0$ such that $E_2 \in U, E_1 \in W, M \subset V$ and $T_t(U) \leq_K T_t(V) \leq_K T_t(W)$ for all $t \geq t_1$. Let $u \in U \cap [E_2, E_2^c]_K \cap C_0$, $w \in W \cap [E_1^c, E_1]_K \cap C_0$ and apply Theorem 2 to conclude that $\omega(u), \omega(w) \subset [E_2^c, E_1^c]_K$. By the inequality above, since $M \subset V$ is invariant, $M \subset [E_2^c, E_1^c]_K$. ∎

Proof of Corollary 4: As B_1 contains no nonempty open set in C_0 then, by the last statement in Theorem 2, $B_1 \cap C_0 = \emptyset$. Similarly, $B_2 \cap C_0 = \emptyset$.

We first show that $\{E_i\}$, $i = 1, 2$, are isolated compact invariant sets. Let U be a neighborhood of E_1 which has empty intersection with $\{E_0\} \cup C_2$ and $[E_2^c, E_1^c]$. If $M \subset U$ is a nonempty compact and invariant set, then $E_0 \notin M$ and so $M \cap C_1 \subset \{E_1\}$ by Proposition 3.1. It follows that $M \setminus \{E_1\} \subset C_0$ and, by Proposition 3.1, $M \subset [E_2, E_1]_K$. If $x \in M \cap C_0$, then $\omega(x) \subset [E_2^c, E_1^c]_K$ by Theorem 2 which, as $\omega(x) \subset M \subset U$ contradicts our choice of U. Thus, $M = \{E_1\}$, establishing our claim. Similarly for E_2.

Let ω be the ω-limit set of a pre-compact forward orbit of an asymptotically autonomous semiflow on C with limit semiflow T. Then $\omega \subset C$ by (H6), it is compact and invariant under T [Th1, Theorem 2.5] and, by Proposition 3.1, $\omega \subset [E_2, E_1]_K$.

If $E_0 \in \omega$, then $\omega = \{E_0\}$. Indeed, by assumption $\{E_0\}$ is an isolated compact invariant set, if $E_0 \in \omega \neq \{E_0\}$, by Lemma 3.1 in [Th1], there exists some $x \in \omega, x \neq E_0$ such that $T_t(x) \to E_0$ as $t \to \infty$, a contradiction to (H2). Therefore, we assume that $E_0 \notin \omega$. Suppose that $E_1 \in \omega \neq \{E_1\}$. Since $\omega \cap C_1$ is a compact invariant set, $\omega \cap C_1 = \{E_1\}$ by Proposition 3.1. Again by Lemma 3.1 in [Th1], there exists some $x \in M, x \neq E_1$, such that $T_t(x) \to E_1, t \to \infty$. Obviously $x \in C_0$, a contradiction to $B_1 \cap C_0 = \emptyset$. Thus, $E_1 \in \omega$ implies $\omega = \{E_1\}$. A similar argument applies to E_2. Therefore we can assume that $E_0, E_1, E_2 \notin \omega$. It follows that $\omega \subset C_0$. Proposition 4.1 implies $\omega \subset [E_2^c, E_1^c]_K$. ∎

Competitive exclusion need not carry over from orbits of T to pre-compact orbits of an asymptotically autonomous semiflow on C with limit semiflow T, but we can conclude that such orbits converge to an equilibrium of T.

Theorem 4.2. *Let the hypotheses of Proposition 3.6 hold. Then every pre-compact orbit of an asymptotically autonomous semiflow on X^+ which has T as limit semiflow and satisfies (H6) converges to E_0, E_1, or E_2.*

We require the following preliminary results concerning the semiflow T which use the hypotheses of Theorem 4.2. For definiteness, we hereafter assume that E_1 attracts all orbits starting in $C_0 \cap [E_2, E_1]_K$.

Lemma 4.3. *Let M in C be non-empty, compact, invariant and $E_0 \notin M$. Then $M \subset (C_0 \cap [E_2, E_1]_K) \cup \{E_1, E_2\}$, and if $E_2 \notin M$ then $M = \{E_1\}$.*

Proof: $M \subset [E_2, E_1]_K$ by Proposition 3.1 and E_i is the only compact invariant set in

C_i. Since $E_0 \notin M$, then $M \cap C_i$ is compact and invariant in C_i; so $M \cap C_i \subset \{E_i\}$, $i = 1, 2$. If M does not contain E_2, then $E_2 <_K M \leq_K E_1$. The strong order preserving property implies that there exists a relatively open set U and some $t_1 \geq 0$ such that $E_2 \in U$ and $T_t(U) \leq_K M = T_t(M)$ for all $t \geq t_1$. There exists $u \in U$ such that $T_t u \to E_1$ as $t \to \infty$. Letting $t \to \infty$ in the above inequality leads to $M = \{E_1\}$. ∎

Lemma 4.4. *The singleton set consisting of E_2 is an isolated compact invariant set for T.*

Proof: Choose a neighborhood U of E_2 which contains no other equilibria and no point of C_1. If M is a nonempty compact invariant subset of U, then $E_0 \notin M$ so $M \cap C_2 \subset \{E_2\}$ by Proposition 3.1. $M \subset [E_2, E_1]_K$, so M can contain no point of C_0 since the orbits of all points in $C_0 \cap [E_2, E_1]_K$ converge to E_1. Thus $M = \{E_2\}$ ∎

Proof of Theorem 4.2: Let ω be the ω-limit set of a pre-compact forward orbit of the asymptotically autonomous semiflow satisfying (H6). Then $\omega \subset C$ is compact and invariant under T by Theorem 2.5 of [Th1]. By Proposition 3.1, $\omega \subset [E_2, E_1]_K$.

Assume that $E_0 \in \omega$, but $\{E_0\} \neq \omega$. Since E_0 is an isolated compact invariant set by Lemma 4.4, by the Butler-McGehee lemma, there exist some $u \in \omega, u \neq E_0$, such that $\omega(u) = \{E_0\}$. But this is impossible by (H2). Thus, $E_0 \in \omega$ implies $\omega = \{E_0\}$. Hereafter, we assume that $E_0 \notin \omega$.

Assume that $E_2 \in \omega$, but $\{E_2\} \neq \omega$. By Proposition 3.1, $\omega \cap C_i \subset \{E_i\}$. Since $\{E_2\}$ is an isolated compact invariant set, the connectedness of ω and the Butler-McGehee lemma implies the existence of $u \in \omega \cap C_0 \cap [E_2, E_1]_K$ such that $\omega(u) = E_2$. But this contradicts that the omega limit set of all such points is E_1. Thus, $E_2 \in \omega$ implies $\omega = \{E_2\}$ so we may assume that $E_2 \notin \omega$. But then by Lemma 4.3, $\omega = \{E_1\}$. ∎

Theorem 4.5. *Let (H0)-(H5) hold and suppose that T is strongly order preserving on $C_0 \cup \{E_1, E_2\}$. Suppose that (a)-(c) of Theorem 1 hold for T. Then the omega limit set ω of any pre-compact orbit of an asymptotically autonomous semiflow satisfying (H6) with limit semiflow T satisfies $\omega = \{E_1\}$ or $\omega = \{E_2\}$, or $\omega \subset S$.*

Proof: $\omega \subset C$ is a compact T-invariant subset of $[E_2, E_1]_K$ by Theorem 2.5 of [Th1] and by Proposition 3.1. To show that $\{E_1\}$ is an isolated compact invariant set, let $x_0 = (E + E_1)/2$, then $x_0 <_K E_1$ so there exists $t_0 \geq 0$ and an open set U of X containing E_1 but no point of $E_0 \cup C_2$ such that $T_t x_0 \leq_K T_t(U \cap (C_0 \cup \{E_1\})$ for $t \geq t_0$. $T_t(x_0) \to E_1$ as $t \to \infty$ by (a) of Theorem 1. If there is a nonempty compact invariant set $M \subset U$, then $M \cap C_1$ is a compact invariant subset of C_1 so it is contained in $\{E_1\}$ by Proposition 3.1. Thus, $M \subset C_0 \cup \{E_1\}$, and consequently $T_t(x_0) \leq_K T_t(M) = M$ for $t \geq t_0$. Letting $t \to \infty$ we have $E_1 \leq_K M$. But $M \subset [E_2, E_1]_K$ by Proposition 3.1, so $M = \{E_1\}$. Thus, $\{E_1\}$ is an isolated compact invariant set for T. Now suppose that $E_1 \in \omega \neq \{E_1\}$. By Lemma 3.1 [Th1], there is a point $w \neq E_1$ and a full T-orbit $\sigma : \mathbb{R} \to \omega$ satisfying $\sigma(0) = w$, $\sigma(t) \in U$ for $t \leq 0$ and $\sigma(t) \to E_1$ as $t \to -\infty$. As $U \subset B_1$, we conclude that $O(w) \subset B_1$. If $O(w) \equiv \{\sigma(t) : t \in \mathbb{R}\} \subset C_1$, then $O(w) \cup \{E_1\}$ is a compact invariant subset of C_1 and hence coincides with $\{E_1\}$ by Proposition 3.1. But this contradicts $w \neq E_1$. Therefore, $O(w)$ must contain a point of C_0 and, since C_0, C_1 are positively invariant, $O(w) \subset C_0$ and hence $T_t(x_0) \leq_K T_t(\sigma(s))$ for $s \leq 0$ and $t \geq t_0$. Consequently, $T_n(x_0) \leq_K w = T_n(\sigma(-n))$ for large n and letting $n \to \infty$, we conclude that $E_1 \leq_K w$, a contradiction to $w \neq E_1$ and $w \in \omega \subset [E_2, E_1]_K$. Consequently, if $E_1 \in \omega$, then $\omega = \{E_1\}$. Indeed, by the invariance of ω, if $\omega \cap B_1 \neq \emptyset$, then $\omega = \{E_1\}$. Similarly, if $\omega \cap B_2 \neq \emptyset$, then $\omega = \{E_2\}$. But if ω contains no point of $B_1 \cup B_2$, then $\omega \subset S$. ∎

5. Discussion

Under the assumptions (H1), (H2), (H3), and (H5), with T being strongly order preserving on the forward invariant set $C_0 \cup \{E_1, E_2\}$ and $\{E_0\}$ an isolated compact invariant set, two species competition can be fairly completely classified according to the local attractivity of the one-species equilibria, E_1, E_2, provided that at most one coexistence equilibrium, $E \in C_0$, exists (Theorem 3.2 for $u_0 = E_1, v_0 = E_2$). When we speak about 'almost all orbits' below, we mean all orbits starting in an open dense subset whose complement intersected with C_0 is unordered and, if the state space is finite dimensional, of Lebesgue measure 0.

Case 1 (Stable Coexistence): E_1 is not locally attractive from below and E_2 is not locally attractive from above.

Then there exists a unique coexistence equilibrium, $E \in C_0$, with open basin of attraction. All orbits converge towards an equilibrium, and under additional assumptions, all orbits in C_0 converge towards E. (See Theorem 2 and Corollary 3 in case $E_1^c = E_2^c$.)

Case 2.1 (Competitive Exclusion): E_1 is not locally attractive from below, but E_2 is locally attractive from above, no coexistence equilibrium.

We have relative competitive exclusion, as all orbits starting in the order interval between the two one-species equilibria converge to E_2. Under additional assumptions, competitive exclusion holds everywhere as all orbits in C_0 converge to E_2. (See Proposition 3.6 and the remark following its proof).

Case 2.2 (Semi-Stability): E_1 is not locally attractive from below, but E_2 is locally attractive from above, unique coexistence equilibrium.

In this degenerate case, all orbits starting in a dense subset of the order interval between the two one-species equilibria converge to either E_2 or E. Under additional assumptions almost all orbits in C_0 converge to either E_2 or E. (See Theorem 3.5 and the remarks subsequent to its proof.)

Cases 3.1, 3.2: E_1 is locally attractive from below, but E_2 is not locally attractive from above.

These cases are symmetric to the Cases 2.1 and 2.2.

Case 4 (Bi-Stability): E_1 is locally attractive from below and E_2 is locally attractive from above.

If no coexistence equilibrium exists, both one-species equilibria have non-empty open basins of attraction and almost all orbits converge to one of them (Proposition 3.7).

If a (unique) coexistence equilibrium, E, exists (which is the case if T_t is condensing on $[E_2, E_1]_K$ for each $t > 0$), this scenario remains unchanged, if E is an interior point of C and the semiflow is C^1 in a neighborhood. An unordered separatrix separates the two basins of attraction. In the non-generic case that the derivative at E has its spectral radius equal to one, it is required to have additional positivity properties, further T_t should be condensing for all $t > 0$. (See Theorem 1, Theorem 3.3 and Theorem 3.4).

In all four cases, almost all orbits converge towards an equilibrium. If the semiflow is asymptotically autonomous rather than autonomous, the classification is far less complete and the convergence properties are inherited from the limit semiflow only in Case 1, Case 2.1 and Case 3.1. (See Corollary 4, Theorem 4.2, and Theorem 4.5.)

We mention in the Introduction that competition models may also lead to discrete systems, either directly or indirectly as time maps of periodic continuous systems (see the results in [HL, HSW2, Ta2]). The main tool in this paper, for strongly order preserving (asymptotically) autonomous semiflows, is the result on totally ordered arcs (Theorem 3.5 in [ST2]) which does not seem to have a counterpart for strongly order preserving maps. It can partially be replaced by the Dancer-Hess result ([DH] or

[He]) on connecting orbits for strictly order-preserving maps, if the image of $[E_2, E_1]_K$ has compact closure under the map. Along these lines, a discrete analog of Proposition 3.6 (i) has been proved in [HSW2], Theorem A (see also [HL], Theorem 1.5), and it is possible to show discrete analogs of our Theorem 2 (cf. [HL],Theorem 1.1). Using asymptotically autonomous maps [Zh], one can obtain discrete versions of Corollary 4 and Theorem 4.2. Discrete analogs of Theorem 1, Theorems 3.3, 3.4, and 3.5, and of Theorem 4.5, however, have been elusive so far.

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