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*Journal of*  
MATHEMATICAL  
ANALYSIS AND  
APPLICATIONS

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# Strong positivity of solutions to parabolic and elliptic equations on nonsmooth domains

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Received 12 March 2001

Submitted by H. Levine

## Abstract

Nonnegativity of weak solutions of parabolic and elliptic equations on nonsmooth domains is established. Strong positivity of weak solutions to elliptic equations is proved via a boundary weak Harnack inequality.

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## 1. Introduction

Maximum principles for parabolic and elliptic equations are an important tool in studying many problems in application, including positivity of steady states of systems of nonlinear elliptic equations and the dynamics of solutions to parabolic systems. In the first, if the system is not variational then the index theory is a promising and powerful tool in proving the existence of solutions. The index theory for nonlinear maps defined on positive cones of Banach spaces (see [1]) has been extensively used to study the existence of positive steady states. One of the fundamental steps to apply this theory is to verify that the map is strongly positive and this usually reduces to the fact that nontrivial and nonnegative solutions

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1 to certain boundary value elliptic problem are in fact strictly positive. Strong 1  
 2 positivity is also a crucial component in establishing the strong monotonicity for 2  
 3 maps if one wishes to apply the theory of monotone dynamical systems to study 3  
 4 the dynamics of solutions. 4

5 In particular, on a bounded open set  $\Omega \subset \mathbb{R}^n$ , we often consider nonnegative 5  
 6 solutions to the elliptic equation 6

$$7 \quad -D_j(A_{ij}(x)D_i u) + B_i(x)D_i u + c(x)u = f(x), \quad \text{in } \Omega \quad 7$$

8 with certain boundary condition on the boundary  $\partial\Omega$  such as Dirichlet or Robin 8  
 9 boundary conditions. Given a solution  $u \neq 0$  and  $u \geq 0$  on  $\Omega$ , we want to 9  
 10 establish that the solution is positive on the closure of  $\Omega$ . If the boundary 10  
 11  $\partial\Omega$  is sufficiently smooth and  $f(x), c(x) \geq 0$  in  $\Omega$  then the strong positivity 11  
 12 of nontrivial nonnegative solutions is just a simple consequence of the Hopf 12  
 13 maximum principle, see Protter and Weinberger [8]. Maximum principles for 13  
 14 parabolic equations can be found in [6]. However, for Lipschitz or even piecewise 14  
 15 smooth boundary domains, the problem becomes more complicated and strong 15  
 16 positivity is no longer true in general. Although there are extensions of the 16  
 17 maximum principle which specify the sign of the directional derivative in outward 17  
 18 pointing directions, even at “edges” or “corners” if the domain is piecewise 18  
 19 smooth, strong regularity hypotheses are required and the conclusion is only 19  
 20 a weak inequality. Serrin [9] seems to be the first to obtain such results, later 20  
 21 extended by Gidas et al. in [3]. See also the edge point lemma in [2]. Most related 21  
 22 to our paper is the lemma on the inner derivative of Nadirashvili [7, Corollary 3.2] 22  
 23 which requires certain boundedness on the boundary data, and therefore cannot 23  
 24 apply to the examples below. 24  
 25

26 The following example (due to A. Castro) is a bit discouraging. 26

27 **Example 1.** Let  $\Omega = (0, \pi/4) \times (0, \pi/4)$  and  $u(x, y) = \sin(x) + \sin(y)$ . Then  $u$  27  
 28 satisfies 28

$$29 \quad \begin{cases} -\Delta u = \sin(x) + \sin(y), & \text{in } \Omega, \\ -u_x(0, y) + \frac{1}{\sin(y)}u(0, y) = 0, & y \in (0, \pi/4), \\ -u_y(x, 0) + \frac{1}{\sin(x)}u(x, 0) = 0, & x \in (0, \pi/4). \end{cases} \quad (1.1) \quad 29$$

30 Obviously  $u(x, y) > 0$  in  $\Omega$  but  $u(0, 0) = 0$ . Thus,  $u$  is nontrivial nonnegative 30  
 31 but not strictly positive. The Hopf boundary point lemma fails at the corner point 31  
 32  $(0, 0)$  where the interior sphere condition is not verified. 32

33 In the same way, we can consider the function  $u(x, y, z) = \sin(x) + \sin(y) +$  33  
 34  $\sin(z)$  defined on the box  $\Omega = [0, \pi/4]^3$  in  $\mathbb{R}^3$ . This function satisfies 34

$$35 \quad \begin{cases} -\Delta u = \sin(x) + \sin(y) + \sin(z), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + b(x, y, z)u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.2) \quad 35$$

1 where, for example,  $b(x, y) = 1/(\sin(x) + \sin(y))$  on the plan  $z = 0$ . Again,  $u$  is  
 2 nonnegative in  $\Omega$  but vanishes at the corner point  $(0, 0, 0)$ .

3  
 4 In the above examples, not only that  $\partial\Omega$  is not smooth but the data  $b$   
 5 is discontinuous and unbounded on  $\partial\Omega$ . However, a simple application of  
 6 Theorem 4 shows that the above are the borderline cases. We assert that

7  
 8 **Example 2.** Let  $\alpha \in (0, 1)$  and  $\Omega = (0, \pi/4)^2$ . Consider the problem

$$\begin{cases} -\Delta u = f(x, y), & \text{in } \Omega, \\ -u_x(0, y) + \frac{1}{\sin^\alpha(y)}u(0, y) = 0, & y \in (0, \pi/4), \\ -u_y(x, 0) + \frac{1}{\sin^\alpha(x)}u(x, 0) = 0, & x \in (0, \pi/4), \end{cases} \quad (1.3)$$

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 13 where  $f(x, y)$  is a bounded nonnegative function on  $\Omega$ . If  $u(x, y)$  is continuous  
 14 on  $\bar{\Omega}$  and solves (1.3) then  $u(0, 0) > 0$ .

15  
 16 A counterpart of (1.2) can be constructed similarly.

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 18 In Section 2, we present a simple proof of nonnegativity of solutions of  
 19 parabolic and elliptic equations. We then prove strong positivity of nontrivial  
 20 nonnegative weak solutions of elliptic equations via boundary Harnack inequality  
 21 in Section 3.

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 23  
 24 **2. Nonnegativity of solutions**

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 26 Let  $\Omega$  be a domain in  $\mathbb{R}^n$  such that the following integration by parts formula

$$\int_{\Omega} \operatorname{div} UV \, dx = \int_{\partial\Omega} \frac{\partial U}{\partial \nu} V \, d\sigma - \int_{\Omega} U \nabla \nabla V \, dx \quad (2.1)$$

27  
 28  
 29 holds for any  $U \in W^{1,2}(\Omega, \mathbb{R}^n)$  and  $V \in W^{1,2}(\Omega)$  that make the above integrals  
 30 finite. The terms in the boundary integral are understood in the sense of traces.

31  
 32 For a function  $u$  in  $W^{1,1}(\Omega)$  we denote the positive and negative parts of  $u$   
 33 respectively by  $u^+$  and  $u^-$ . That is,  $u = u^+ + u^-$ ,  $u^+ \geq 0$  and  $u^- \leq 0$ . By [4,  
 34 Lemma 7.6, p. 152], we have that

$$Du^+ = \begin{cases} Du, & \text{if } u > 0, \\ 0, & \text{if } u \leq 0, \end{cases} \quad Du^- = \begin{cases} Du, & \text{if } u < 0, \\ 0, & \text{if } u \geq 0. \end{cases} \quad (2.2)$$

35  
 36  
 37 It follows that, for any indices  $i, j$ ,

$$u^+ u^- = D_i u^+ D_j u^- = D_i u^+ u^- = u^+ D_i u^- = 0, \quad \text{a.e. in } \Omega. \quad (2.3)$$

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 39  
 40 For some  $T > 0$ , let  $Q_T = \Omega \times (0, T)$ . We consider the following inequality  
 41 in  $Q_T$ .

$$u_t - Lu + c(x, t)u \geq 0, \quad (2.4)$$

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1 where

$$2 \quad Lu = D_j(A_{ij}(x, t)D_i u) + B_i(x, t)D_i u, \quad 2$$

3 and  $B_i(x, t), c(x, t)$  are bounded measurable functions on  $Q_T$ . Moreover,  $4$   
 $L$  satisfies the following uniform ellipticity condition for some  $d > 0$ .  $5$

$$6 \quad A_{ij}(x, t)\xi_i\xi_j \geq d|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, (x, t) \in Q_T. \quad 6 \quad (2.5)$$

7 We also consider the following inequality on  $\partial\Omega \times (0, T)$ .  $7$

$$8 \quad A_{ij}D_i u \mathbf{n}_j + b(x, t)u \geq 0, \quad 8 \quad (2.6)$$

9 where  $b(x, t) \geq 0$  is a given function in  $L^\infty([0, T], L^2(\partial\Omega))$  and  $v = (\mathbf{n}_j)$ .  $9$

10 Formally multiplication (2.4) (following [5, Chapter 3]) by a nonnegative  $10$   
 smooth function  $\eta$  and integration by parts lead us to the following definition.  $11$   
 Let  $V_2(Q_T)$  (see [5, p. 6]) be the Banach space consisting of all functions in  $12$   
 $W_2^{1,0}(Q_T)$  having finite norm  $13$

$$14 \quad \|u\|_{V_2(Q_T)} = \sup_{0 \leq t \leq T} \|u(\bullet, t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)}. \quad 14$$

15 A function  $u \in V_2(Q_T)$  is said to satisfy weakly (2.4) and (2.6) if for any  $15$   
 nonnegative  $\eta \in W_2^{1,1}(Q_T)$   $16$

$$17 \quad \int_{\Omega} u(x, t)\eta(x, t) dx - \int_{\Omega} u(x, 0)\eta(x, 0) dx - \iint_{Q_t} u\eta_t dx dt \quad 17$$

$$18 \quad + \iint_{Q_t} A_{ij}D_i u D_j \eta - B_i D_i u \eta + c u \eta dx dt \geq - \int_0^t \int_{\partial\Omega} b u \eta d\sigma dt, \quad 18 \quad (2.7)$$

19 for almost every  $t \in (0, T)$ . Here,  $Q_t = \Omega \times (0, t)$ .  $19$

20 It is easy to see that if  $u \in C^{2,1}(\overline{Q_T})$  satisfies (2.4), (2.6) pointwise then (2.7)  $20$   
 holds for  $u$ . Conversely, if  $u \in V_2(Q_T)$  and satisfies (2.7) and  $u$  is sufficiently  $21$   
 smooth then it also verifies (2.4), (2.6) pointwise.  $22$

23 We will now prove that  $23$

24 **Theorem 1.** Let  $u(x, t)$  be in  $V_2(Q_T)$  and satisfy (2.4) and (2.6) weakly. If  $24$   
 $u(x, 0) \geq 0$  then  $u(x, t) \geq 0$  a.e. in  $Q_T$ .  $25$

26 **Proof.** First of all, by the change of variables  $u = e^{kt}U$ , we can replace the  $26$   
 coefficient  $c$  in (2.4) by  $c + k$  and prove the assertion of the lemma for  $U$ .  $27$   
 Therefore, we will assume that  $u$  satisfies  $28$

$$29 \quad u_t - Lu + (c + k)u \geq 0. \quad 29 \quad (2.8)$$

30 Using the Steklov average and taking to the limit (see [5, Chapter 3]) we can  $30$   
 formally take  $\eta = -u^- \geq 0$  in (2.7) to obtain  $31$

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega} (u^-(x, t))^2 dx - \frac{1}{2} \int_{\Omega} (u^-(x, 0))^2 dx \\
 & + \iint_{Q_t} (A_{ij} D_i u D_j u^- - B_i D_i u u^- + (c+k) u u^-) dx dt \\
 & \leq - \int_0^t \int_{\partial\Omega} b u u^- d\sigma dt.
 \end{aligned} \tag{2.9}$$

Because  $D_i u D_j u^- = D_i u^- D_j u^-$  and  $u u^- = (u^-)^2$  a.e. by (2.3), and  $u^-(x, 0) \equiv 0, b \geq 0$  by our assumption, we derive from the above that

$$\begin{aligned}
 & \int_{\Omega} \frac{1}{2} (u^-(x, t))^2 dx \\
 & + \iint_{Q_t} (A_{ij} D_i u^- D_j u^- - B_i D_i u^- u^- + (c+k)(u^-)^2) dx dt \leq 0.
 \end{aligned} \tag{2.10}$$

For any  $\varepsilon > 0$ , there exists  $C(\varepsilon)$  such that,

$$|B_i D_i u u^-| = |B_i D_i u^- u^-| \leq \varepsilon |D u^-|^2 + C(\varepsilon) (u^-)^2. \tag{2.11}$$

Using the above in (2.10) with  $\varepsilon = d/2$  and taking into account (2.5), we obtain

$$\begin{aligned}
 & \int_{\Omega} \frac{1}{2} (u^-(x, t))^2 dx + \frac{d}{2} \iint_{Q_t} |D(u^-)|^2 dx dt \\
 & \leq \iint_{Q_t} [C(\varepsilon) - (c+k)] (u^-)^2 dx dt.
 \end{aligned}$$

Next, we choose  $k$  sufficiently large such that  $C(d/2) - (c+k) \leq 0$ . The above yields

$$\begin{aligned}
 & \int_{\Omega} \frac{1}{2} (u^-(x, t))^2 dx + \frac{d}{2} \iint_{Q_t} |D(u^-)|^2 dx dt \leq 0, \\
 & \text{for a.e. } t \in (0, T).
 \end{aligned} \tag{2.12}$$

This implies that  $\int_{\Omega} (u^-)^2(x, t) dx = 0$  and therefore  $u^-(x, t) \equiv 0$ . We then conclude that  $u(x, t) \geq 0$ , for a.e.  $t \in (0, T)$ .  $\square$

An elliptic version of Theorem 1 is also available. We say that a function  $u \in W^{1,2}(\Omega)$  weakly satisfies

$$\begin{aligned}
 & -D_j (A_{ij} D_i u) + B_i D_i u + cu \geq 0, \quad \text{in } \Omega, \\
 & A_{ij} D_i u n_j + b(x)u \geq 0, \quad \text{on } \partial\Omega
 \end{aligned} \tag{2.13}$$

1 if, for any nonnegative  $\phi \in W^{1,2}(\Omega)$ , we have 1

$$\int_{\Omega} (A_{ij} D_i u D_j \phi + B_i D_i u \phi + cu\phi) dx \geq - \int_{\partial\Omega} bu\phi d\sigma. \quad (2.14)$$

2 We assume the following conditions on (2.13). For some  $d > 0$ , 2

$$A_{ij}(x)\xi_i\xi_j \geq d|\xi|^2, \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n. \quad (2.15)$$

3 We suppose furthermore that  $-\frac{1}{2}D_i B_i + c$  is weakly nonnegative in the 3  
 4 following sense 4

$$\int_{\Omega} \left( \frac{1}{2} B_i D_i v + cv \right) dx \geq 0, \quad \forall v \in W^{1,1}(\Omega), v \geq 0. \quad (2.16)$$

5 We then have the following result. 5

6 **Theorem 2.** Assume (2.15), (2.16) and 6

$$\int_{\Omega} c dx + \int_{\partial\Omega} b d\sigma > 0.$$

7 If  $u \in W^{1,2}(\Omega)$  satisfies (2.13) weakly then  $u(x) \geq 0$  a.e. 7

8 **Proof.** We follow the argument of the proof of Theorem 1. Note that we can no 8  
 9 longer replace  $c$  by  $c + k$  here. Using  $\phi = u^-$  in (2.14), we obtain 9

$$d \int_{\Omega} |Du^-|^2 dx + \int_{\Omega} (B_i D_i u^- u^- + c(u^-)^2) dx \leq - \int_{\partial\Omega} b(u^-)^2 d\sigma.$$

10 Since  $(u^-)^2 \in W^{1,1}(\Omega)$  and  $D((u^-)^2) = 2Du^- u^-$ , the second integral is 10  
 11 nonnegative by (2.16). We conclude that  $Du^- \equiv 0$  and therefore  $u^-$  is a constant. 11

12 The above reduces to 12

$$(u^-)^2 \left[ \int_{\Omega} c dx + \int_{\partial\Omega} b d\sigma \right] = \int_{\Omega} c(u^-)^2 dx + \int_{\partial\Omega} b(u^-)^2 d\sigma \leq 0.$$

13 Thanks to the assumption on  $b, c$ , the above implies  $u^- \equiv 0$  and concludes our 13  
 14 proof.  $\square$  14

15 **Remark 2.1.** If  $B_i$ 's are differentiable, (2.16) implies that  $-\frac{1}{2}D_i B_i + c \geq 0$  a.e. 15  
 16 in  $\Omega$ . To see this, one needs only to take  $v \in W^{1,1}(\Omega)$  with compact support 16  
 17 and use integration by parts. Taking constant  $v$  in (2.16), one also derives that 17  
 18  $\int_{\Omega} c dx \geq 0$ . Therefore, we could as well assume that  $\int_{\partial\Omega} b d\sigma > 0$  in the above 18  
 19 theorem. Finally, by a simple use of Young's inequality as in (2.11) of the proof 19  
 20 20

of Theorem 1, we can obtain the above conclusion when the assumption on  $B_i$  and  $c$  in (2.16) is replaced by

$$4dc(x) - \sum_i |B_i(x)|^2 > 0, \quad \forall x \in \Omega.$$

The proof is similar so that it will be omitted.

**Remark 2.2.** The above arguments can apply equally to mixed boundary condition problems. In fact, we can replace (2.6) by

$$\begin{aligned} A_{ij}D_i u_n_j + b(x, t)u &\geq 0, \quad \text{on } \Gamma_1 \times (0, T), \\ u &\geq 0, \quad \text{on } \Gamma_2 \times (0, T), \end{aligned} \tag{2.17}$$

where  $\Gamma_1, \Gamma_2$  are open subsets of  $\partial\Omega$  and  $\partial\Omega = \Gamma_1 \cup \Gamma_2$ . To see this, we notice that  $u^- = 0$  on  $\Gamma_2 \times (0, T)$ .

The conclusions of Theorems 1 and 2 hold as well for semilinear parabolic and elliptic equations. For example, we can consider the following problem

$$\begin{aligned} u_t - D_j(A_{ij}D_i u) + B_i D_i u + c(x, t)u &= f(x, t, u), \quad (x, t) \in Q_T \\ A_{ij}D_i u_n_j + b(x, t)u &= g(x, t), \quad (x, t) \in \partial\Omega \times (0, T). \end{aligned} \tag{2.18}$$

We then assume that

$$f(x, t, u) \geq 0, \quad \text{for } u \geq 0 \quad \text{and} \quad f(x, t, u) \leq 0, \quad \text{for } u < 0. \tag{2.19}$$

Under these conditions,  $f(x, t, u(x, t))(-u^-(x, t)) \leq 0$  for any given function  $u$ . Therefore, we can obtain (2.9) in the proof of Theorem 1 for the system (2.18) and follow the same argument there to prove the following result.

**Theorem 3.** Let  $u(x, t)$  be in  $V_2(Q_T)$  and satisfy (2.18) weakly. If  $g(x, t) \geq 0$  and  $u(x, 0) \geq 0$  then  $u(x, t) \geq 0$  in  $Q_T$  a.e.

### 3. Strong positivity

In this section, we will show that *nontrivial* nonnegative weak solutions of the elliptic equation

$$\begin{aligned} -D_j(A_{ij}D_i u) + B_i D_i u + cu &= f(x), \quad \text{in } \Omega, \\ A_{ij}D_i u_n_j + b(x)u &= g(x), \quad \text{on } \partial\Omega \end{aligned} \tag{3.1}$$

are strictly positive on  $\bar{\Omega}$  if  $c, f, g$  are nonnegative functions. We discuss below the assumptions will be made on the data of (3.1) and the domain  $\Omega$ .

Let  $B(x_0, R)$  denote the ball in  $\mathbb{R}^n$  that centers at  $x_0$  with radius  $R$ . For any given  $x_0 \in \bar{\Omega}$  and  $R > 0$  we write  $\Omega(x_0, R) = \Omega \cap B(x_0, R)$  and  $\partial\Omega_R(x_0) =$

1  $\partial\Omega \cap B(x_0, R)$ . If  $x_0$  is understood, we simply write  $\Omega(R)$ ,  $\partial\Omega_R$  for  $\Omega(x_0, R)$ , 1  
 2  $\partial\Omega_R(x_0)$ . 2

3 We impose the following conditions on the geometry of  $\Omega$  and the boundary 3  
 4  $\partial\Omega$ : For all  $x_0 \in \bar{\Omega}$ ,  $R > 0$  the set  $\Omega(x_0, R)$  is convex and there exist positive 4  
 5 constants  $\alpha_1, \alpha_2, \alpha_3$  such that 5

6 
$$\alpha_1 R^n \leq |\Omega(x_0, R)| \leq \alpha_2 R^n, \quad \sigma(\partial\Omega \cap B(x_0, R)) \leq \alpha_3 R^{n-1}, \quad (3.2)$$
 6

7 where  $|A|$ ,  $\sigma(A)$  are respectively the  $n$ -dimensional and the surface measures of 7  
 8 a set  $A$ . Furthermore,  $\partial\Omega$  is assumed to be piecewisely smooth. 8  
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10 Regarding the coefficients of the equation, we impose the following conditions. 10

11 (A.1) (Ellipticity) There exist positive constants  $d_1, d_2$  such that for all real 11  
 12 vectors  $\xi = (\xi_i) \in \mathbb{R}^n$  we have 12

13 
$$d_1 |\xi|^2 \leq A_{ij}(x) \xi_i \xi_j \leq d_2 |\xi|^2, \quad \text{a.e. in } \Omega.$$
 13

14 (A.2)  $b \in L^{q_0}_{loc}(\partial\Omega)$  for some  $q_0 > n - 1$ . 14  
 15

16 (A.3)  $B_i, c, f \in L^\infty_{loc}(\Omega)$ . 16  
 17

18 The main result of this work is the following 18  
 19

20 **Theorem 4.** Assume that  $f$  and  $g$  are nonnegative functions. Let  $u \in W^{1,2}(\Omega) \cap$  20  
 21  $L^\infty(\Omega)$  be a nontrivial nonnegative solution to (3.1). Then  $u$  is Hölder 21  
 22 continuous on  $\bar{\Omega}$  with some  $\alpha > 0$ . Moreover,  $u$  is strictly positive on  $\bar{\Omega}$ . That 22  
 23 is, there exists  $C_0 > 0$  such that 23

24 
$$u(x) \geq C_0, \quad \forall x \in \bar{\Omega}.$$
 24  
 25

26 We remark here that if  $b \geq 0$  and (2.16) holds then solutions to (3.1) are 26  
 27 nonnegative by Theorem 2. 27

28 The proof of Theorem 4 is based on the weak Harnack inequality which will 28  
 29 be extended up to the boundary for nonnegative weak supersolutions of (3.1). We 29  
 30 recall that a weak supersolution  $u$  of (3.1) is a function in  $W^{1,2}(\Omega)$  and weakly 30  
 31 satisfies 31  
 32

33 
$$\begin{aligned} -D_j(A_{ij} D_i u) + B_i D_i u + cu &\geq 0, \quad \text{in } \Omega, \\ A_{ij} D_i u n_j + b(x)u &\geq 0, \quad \text{on } \partial\Omega. \end{aligned} \quad (3.3)$$
 33  
 34  
 35

36 That is, for any nonnegative  $\phi \in W^{1,2}(\Omega)$ , we have 36

37 
$$\int_{\Omega} (A_{ij} D_i u D_j \phi + B_i D_i u \phi + cu\phi) dx \geq - \int_{\partial\Omega} bu\phi d\sigma. \quad (3.4)$$
 37  
 38  
 39

40 We collect here some known facts that will be needed in our proof. First, as 40  
 41  $\partial\Omega$  is piecewisely smooth, we have the following inequality from [5, p. 69] 41

42 
$$\|u\|_{L^q(\partial\Omega)} \leq \|Du\|_{L^2(\Omega)}^\alpha \|u\|_{L^2(\Omega)}^{1-\alpha}, \quad \alpha = \frac{n}{2} - \frac{n-1}{q}, \quad (3.5)$$
 42  
 43

1 for all  $u \in W^{1,2}(\Omega)$  that satisfies  $u_\Omega = \int_\Omega u \, dx = 0$  and all  $q \in [2(n-1)/n,$  1  
 2  $2(n-1)/(n-2)]$  and  $q \in [1, \infty)$  for  $n = 2$ . If  $q < 2(n-1)/(n-2)$  then  $\alpha < 1$ . 2  
 3 Therefore, for any  $u \in W^{1,2}(\Omega)$ , we can apply the above inequality to  $u - u_\Omega$  3  
 4 and then Young's inequality to get 4

$$5 \quad \|u\|_{L^q(\partial\Omega)}^2 \leq \varepsilon \|Du\|_{L^2(\Omega)}^2 + C(\varepsilon) \|u\|_{L^2(\Omega)}^2, \quad 5$$

$$6 \quad \forall \varepsilon > 0, q \in \left[ \frac{2(n-1)}{n}, \frac{2(n-1)}{n-2} \right). \quad 6$$

7 (3.6) 7  
 8 8  
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10 We should remark that, as a consequence of (A.2), Hölder inequality and (3.5), 10  
 11 the boundary integral in (3.4) is finite for all  $u, \phi \in W^{1,2}(\Omega)$ . 11

12 We will also need the following result by John and Nirenberg [4, Theo- 12  
 13 rem 7.21]. 13

14 **Lemma 3.1.** *Let  $u \in W^{1,1}(\Omega)$  where  $\Omega$  is convex, and suppose there exists a 15  
 16 constant  $K$  such that 16*

$$17 \quad \int_{\Omega(x_0, R)} |Du| \, dx \leq K R^{n-1}, \quad \text{for all } x_0 \text{ and } R > 0. \quad 17$$

18 18  
 19 19  
 20 20

21 Then there exists positive constants  $\sigma_0, C$  depending only on  $n$  such that 21

$$22 \quad \int_{\Omega} \exp\left(\frac{\sigma}{K} |u - u_\Omega|\right) \, dx \leq C (\text{diam}(\Omega))^n, \quad 22$$

23 23  
 24 24

25 where  $\sigma = \sigma_0 |\Omega| (\text{diam}(\Omega))^{-n}$  and  $u_\Omega = \int_\Omega u \, dx$ . 25  
 26 26

27 We then prove the following weak Harnack inequality for (3.3). 27  
 28 28

29 **Lemma 3.2.** *Let  $u$  be a nonnegative function in  $W^{1,2}(\Omega) \cap L^\infty(\Omega)$  that 29  
 30 satisfies (3.3) weakly. Then for any  $\lambda_0 \in [1, n/(n-2))$  there is  $C > 0$  such that 30  
 31 31*

$$32 \quad \left( \int_{\Omega(x_0, 2R)} u^{\lambda_0} \, dx \right)^{1/\lambda_0} \leq C \text{essinf}_{\Omega(x_0, R)} u, \quad 32$$

33 33  
 34 34  
 35 35

36 for any  $x_0 \in \bar{\Omega}$  and  $R > 0$ . Therefore, if the left-hand side of (3.7) is nonzero for 36  
 37 all  $x_0 \in \bar{\Omega}$  then  $u$  is strictly positive on  $\bar{\Omega}$ . 37  
 38 38

39 **Proof.** We fix an  $x_0 \in \bar{\Omega}$  and  $R > 0$ . For a given  $\epsilon > 0$  we define  $v = u + \epsilon$ . Let 39  
 40  $0 < r' < r \leq 2R$  and  $\eta$  be a  $C^\infty$  cutoff function such that  $\eta \equiv 1$  in  $B(x_0, r')$  and 40  
 41  $\eta \equiv 0$  outside  $B(x_0, r)$  and, furthermore,  $|D\eta| \leq 2/(r - r')$ . 41

42 For  $p < 0$  we set  $\phi = v^p \eta^2$ , which is a legitimate test function, in (3.4) to 42  
 43 obtain 43

$$\int_{\Omega} A_{ij} (pD_i u D_j v v^{p-1} \eta^2 + 2D_i u v^p \eta D_j \eta) + B_i D_i u v^p \eta^2 + c u v^p \eta^2 dx$$

$$\geq - \int_{\partial \Omega} b u v^p \eta^2 d\sigma,$$

or

$$|p| \int_{\Omega} A_{ij} D_i u D_j v v^{p-1} \eta^2 dx$$

$$\leq \int_{\Omega} (2D_i u v^p \eta D_j \eta + B_i D_i u v^p \eta^2 + c u v^p \eta^2) dx + \int_{\partial \Omega} b u v^p \eta^2 d\sigma. \quad (3.8)$$

Assume first that  $p \neq -1$ . Using (A.1), the facts that  $Du = Dv$ ,  $p < 0$ , and the Young inequality we have

$$|D(v^{(p+1)/2})|^2 = \left(\frac{p+1}{2}\right)^2 |Dv|^2 v^{p-1} \leq \frac{1}{d_1} \left(\frac{p+1}{2}\right)^2 A_{ij} D_i u D_j v v^{p-1},$$

$$v^p D_i u \eta D_j \eta = [v^{(p-1)/2} D_i u \eta] [v^{(p+1)/2} D_j \eta]$$

$$\leq \varepsilon v^{p-1} |Dv|^2 \eta^2 + C(\varepsilon) v^{p+1} |D\eta|^2,$$

$$v^p D_i u \eta^2 = [v^{(p-1)/2} D_i u \eta] [v^{(p+1)/2} \eta] \leq \varepsilon v^{p-1} |Dv|^2 \eta^2 + C(\varepsilon) v^{p+1} \eta^2.$$

Choosing  $\varepsilon$  small we derive

$$\int_{\Omega} |D(v^{(p+1)/2})|^2 \eta^2 dx$$

$$\leq C \frac{(p+1)^2}{|p|} \left\{ \int_{\Omega} v^{p+1} (\eta^2 + |D\eta|^2) dx + \int_{\partial \Omega} b u v^p \eta^2 d\sigma \right\},$$

or

$$\int_{\Omega} |D(v^{(p+1)/2} \eta)|^2 dx$$

$$\leq C \frac{(p+1)^2}{|p|} \left\{ \int_{\Omega} v^{p+1} (\eta^2 + |D\eta|^2) dx + \int_{\partial \Omega} b u v^p \eta^2 d\sigma \right\}. \quad (3.9)$$

If  $B(x_0, r) \subset \Omega$  the boundary integral is zero. Otherwise, since  $u < v$ , we use Hölder inequality to get

$$\int_{\partial \Omega} b u v^p \eta^2 d\sigma \leq \int_{\partial \Omega} b v^{p+1} \eta^2 d\sigma \leq \|b\|_{L^{q_0}(\partial \Omega_r)} \|v^{p+1} \eta^2\|_{L^q(\partial \Omega)}.$$

Here  $1 < q = q_0/(q_0 - 1) < (n - 1)/(n - 2)$ . Applying (3.6) to  $V = v^{(p+1)/2}\eta$  we have

$$\begin{aligned} \|v^{p+1}\eta^2\|_{L^q(\partial\Omega)} &= \|V\|_{L^{2q}(\partial\Omega)}^2 \\ &\leq \varepsilon \|DV\|_{L^2(\Omega)}^2 + C(\varepsilon)\|V\|_{L^2(\Omega)}^2, \quad \forall \varepsilon > 0. \end{aligned}$$

Hence,

$$\int_{\partial\Omega} bu v^p \eta^2 d\sigma \leq \varepsilon \int_{\Omega} |D(v^{(p+1)/2}\eta)|^2 dx + C(\varepsilon) \int_{\Omega} v^{p+1} \eta^2 dx. \quad (3.10)$$

Thus, for adequately small  $\varepsilon$ , we infer from (3.9) and (3.10) that

$$\int_{\Omega} (|D(v^{(p+1)/2}\eta)|^2 + v^{p+1}\eta^2) dx \leq C \frac{(p+1)^2}{|p|} \int_{\Omega} v^{p+1}(\eta^2 + |D\eta|^2) dx.$$

We can now follow Moser's iteration argument. We first recall the imbedding inequality

$$\|U\|_{L^q(\Omega)} \leq C \|U\|_{W^{1,2}(\Omega)} |\Omega|^{1/q-1/2+1/n}, \quad 1 \leq q \leq \frac{2n}{n-2}.$$

Now choose  $\kappa$  such that  $1 < \kappa \leq n/(n-2)$  and apply the above to  $U = v^{(p+1)/2}\eta$  in  $\Omega(r)$  and  $q = 2\kappa$  we obtain

$$\begin{aligned} &\left( \int_{\Omega} (v^{p+1}\eta^2)^\kappa dx \right)^{1/\kappa} \\ &\leq C |\Omega(r)|^{1/\kappa-1+2/n} \int_{\Omega} (|D(v^{(p+1)/2}\eta)|^2 + v^{p+1}\eta^2) dx \\ &\leq C |\Omega(r)|^{1/\kappa-1+2/n} \frac{(p+1)^2}{|p|} \int_{\Omega} v^{p+1}(|D\eta|^2 + \eta^2) dx. \end{aligned}$$

Let  $\lambda > 0$ . For  $i = 0, \dots, k$ , let  $r_i = R + 2^{-i+1}R$  and  $p_i < 0$  such that  $p_i + 1 = -\lambda\kappa^i$ . We take  $r' = r_{i+1}$  and  $r = r_i$  and  $p = p_i$  in the above inequality. Notice that  $|\eta|, |D\eta| \leq 2^{i+2}/R$ ,  $|\Omega(r_i)| \sim |\Omega(R)| \sim R^n$ ,  $|p| > 1$ . We then obtain from the above inequality that

$$\left( \int_{\Omega(r_{i+1})} v^{-\lambda\kappa^{i+1}} dx \right)^{1/\kappa} \leq C |\Omega(R)|^{1/\kappa-1} 4^i (\lambda\kappa^i)^2 \int_{\Omega(r_i)} v^{-\lambda\kappa^i} dx.$$

Dividing by  $|\Omega(R)|^{1/\kappa}$  and raising both sides to the power  $1/(\lambda\kappa^i)$ , we get

$$\left( \int_{\Omega(r_{i+1})} (v^{-1})^{\lambda\kappa^{i+1}} dx \right)^{1/(\lambda\kappa^{i+1})} \leq C_1^{\kappa^{-i}} C_2^{\kappa^{-i}} \left( \int_{\Omega(r_i)} (v^{-1})^{\lambda\kappa^i} dx \right)^{1/\lambda\kappa^i}$$

with  $C_1 = (C\lambda^2)^{1/\lambda}$  and  $C_2 = (4\kappa^2)^{1/\lambda}$ . Iterating the above gives

$$\left( \int_{\Omega(r_{k+1})} (v^{-1})^{\lambda\kappa^{k+1}} dx \right)^{1/\lambda\kappa^{k+1}} \leq C_1^{\sum_{i=0}^k \kappa^{-i}} C_2^{\sum_{i=0}^k i\kappa^{-i}} \left( \int_{\Omega(3R)} (v^{-1})^\lambda dx \right)^{1/\lambda}. \tag{3.11}$$

As the series in the exponents converge, we now let  $k$  tend to  $\infty$  to obtain

$$\text{esssup}_{\Omega(R)} \frac{1}{v} \leq C \left( \int_{\Omega(3R)} v^{-\lambda} dx \right)^{1/\lambda}. \tag{3.12}$$

Notice that the above holds for all  $\lambda > 0$ .

We now take  $p = -1$  and  $r' = R$ ,  $r = 2R$  for some  $R \in (0, 1)$ . Using (A.1) again, we derive from (3.8) the following.

$$\int_{\Omega} |Dv|^2 v^{-2} \eta^2 dx \leq C \left[ \int_{\Omega} \eta^2 + |D\eta|^2 dx + \int_{\partial\Omega} buv^{-1} \eta^2 d\sigma \right]. \tag{3.13}$$

Using the fact that  $u < v$  and Hölder inequality, we get

$$\int_{\partial\Omega} buv^{-1} \eta^2 d\sigma \leq \int_{\partial\Omega} b\eta^2 d\sigma \leq \|b\|_{L^{q_0}(\partial\Omega_{3R})} (\sigma(\partial\Omega_{3R}))^{1/q},$$

where  $1/q = 1 - 1/q_0 > (n-2)/(n-1)$ . By (3.2),  $(\sigma(\partial\Omega_{3R}))^{1/q} \leq CR^{n-2}$  since  $R < 1$ . On the hand, we have  $|D\eta| \leq 2/R$  and  $R^n \leq R^{n-2}$ . Therefore, we derive from (3.13) the following.

$$\int_{\Omega(R)} |D(\ln v)|^2 dx \leq \int_{\Omega} |Dv|^2 v^{-2} \eta^2 dx \leq CR^{n-2}, \quad \forall R \in (0, 1).$$

The above holds trivially for  $R \geq 1$ . Let  $w = \ln v$ , we have

$$\begin{aligned} \int_{\Omega(R)} |Dw| dx &\leq CR^{n/2} \left( \int_{\Omega(R)} |D(\ln v)|^2 dx \right)^{1/2} \leq CR^{n/2} R^{(n-2)/2} \\ &= CR^{n-1}. \end{aligned}$$

We now use Lemma 3.1 [4, Theorem 7.21] applying to  $\Omega(3R)$ . As  $\text{diam}(\Omega(3R)) \sim R$ ,  $|\Omega(3R)| \sim R^n$  we can find  $\Lambda_0 > 0$  and a constant  $C$  such that for any  $\lambda_0 \in (0, \Lambda_0]$  we have

$$\int_{\Omega(3R)} \exp(\lambda_0 |w - w_0|) dx \leq CR^n,$$

1 where  $w_0 = \int_{\Omega(3R)} w \, dx$ . Therefore, 1

$$\begin{aligned}
 \int_{\Omega(3R)} v^{\lambda_0} \, dx \int_{\Omega(3R)} v^{-\lambda_0} \, dx &= \int_{\Omega(3R)} e^{w\lambda_0} \, dx \int_{\Omega(3R)} e^{-w\lambda_0} \, dx \\
 &= \int_{\Omega(3R)} e^{(w-w_0)\lambda_0} \, dx \int_{\Omega(3R)} e^{(w_0-w)\lambda_0} \, dx \\
 &\leq \left( \int_{\Omega(3R)} e^{|w-w_0|\lambda_0} \, dx \right)^2 \leq CR^{2n}.
 \end{aligned}$$

2 This implies (as  $|\Omega(3R)| \sim R^n$ ) 2

$$\int_{\Omega(3R)} v^{-\lambda_0} \, dx \leq C \left( \int_{\Omega(3R)} v^{\lambda_0} \, dx \right)^{-1}.$$

3 Take  $\lambda = \lambda_0$  in (3.12), we obtain 3

$$\operatorname{esssup}_{\Omega(R)} \frac{1}{v} \leq C \left( \int_{\Omega(3R)} v^{\lambda_0} \, dx \right)^{-1/\lambda_0}. \tag{3.14}$$

4 Let  $\epsilon$  in the definition of  $v$  tend to zero,  $v \rightarrow u$ . We obtain (3.7) from (3.14) for 4  
 all  $\lambda_0 \in (0, \Lambda_0]$ . Finally, we observe that the argument leads to (3.11) continues to 5  
 hold as long as  $p_i < 0$  and  $p_i \neq -1$ . Thus, we can take  $p_i = -1 + \lambda_0 \kappa^i$  provided 6  
 that  $\lambda_0 \kappa^i < 1$  or  $\lambda_0 \kappa^{i+1} < \kappa \leq n/(n-2)$ . We also redefine  $r_i = 2R + 2^{-i}R$ . 7  
 The iteration argument then shows that for any  $\lambda \in [1, n/(n-2))$  we can find 8  
 $\lambda_0 \in (0, \Lambda_0)$  such that 9

$$\left( \int_{\Omega(2R)} v^\lambda \, dx \right)^{1/\lambda} \leq C \left( \int_{\Omega(3R)} v^{\lambda_0} \, dx \right)^{1/\lambda_0}.$$

10 This and (3.14) conclude our proof.  $\square$  10

11 We now give the 11

12 **Proof of Theorem 4.** Once such weak Harnack inequality was proven for (3.3), 12  
 we can follow exactly the lines in [4, Theorem 8.22, p. 200] to show that bounded 13  
 weak solutions of (3.1) are Hölder continuous with some Hölder exponent  $\alpha > 0$ . 14  
 As  $f, g \geq 0$ ,  $u$  is also a weak supersolution so that (3.7) holds for  $u$ . Moreover, 15  
 because  $u$  is continuous, the essential infimum in (3.7) is actually the infimum. 16  
 Also, a similar argument to that of [4, Theorem 8.19] shows that  $u$  cannot vanish 17  
 in the interior of  $\Omega$ . This implies that the left-hand side of (3.7) is nonzero and 18  
 therefore  $u$  is strictly positive on  $\partial\Omega$ .  $\square$  19

1 Finally, it is easy to see that the data  $b$  in Example 2 discussed in Section 1 is in 1  
 2  $L_{loc}^{q_0}$  for some  $q_0 > 1$ . Here,  $n = 2$  so that the claim there follows from Theorem 4. 2  
 3 3  
 4 4

## 5 References 5

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