Numerical Methods for ODEs & PDEs

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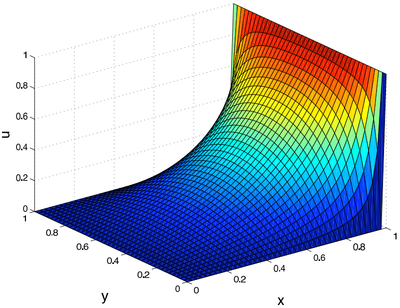


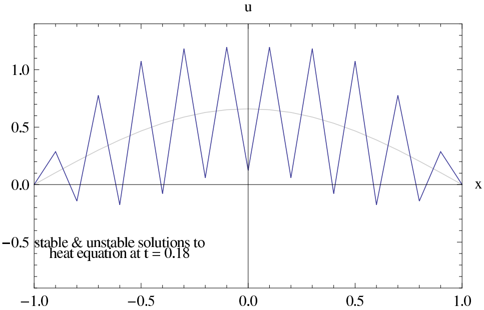
Equivalence Theorem (Lax-Richtmyer)

For *consistent* numerical approximations, *stability* & *convergence* ($||u(t_n) - u_n|| \to 0$) are equivalent.

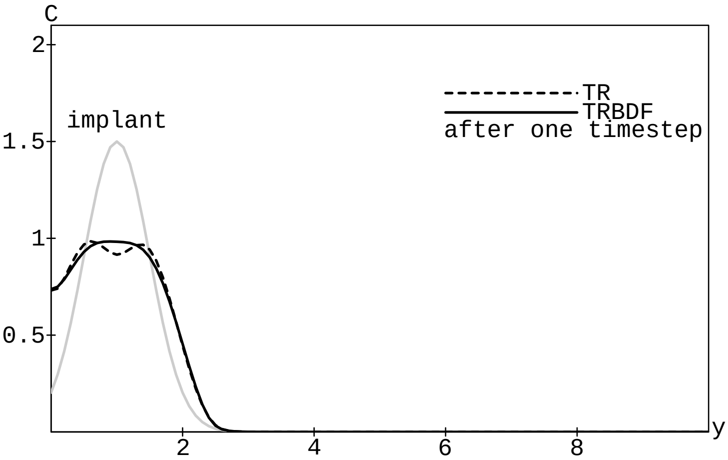
Consistency

$$\frac{u_{n+1} - u_n}{\Delta t} = \frac{du}{dt} + \frac{\Delta t}{2}\frac{d^2u}{dt^2} + \cdots$$

Instability



A-Stability & L-Stability



Derivatives & Notation

Second-order accurate central difference approx. to first derivative

$$\left(\frac{df}{dx}\right)_i \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x}$$

Second-order accurate central difference approx. to second derivative

$$\left(\frac{d^2f}{dx^2}\right)_i \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2}$$

First-order accurate one-sided difference approx. to first derivative

$$\frac{du}{dt} \approx \frac{u_{n+1} - u_n}{\Delta t}$$

Taylor Series

To verify formulas, use Taylor series

$$f_{i\pm1} = f(x_i \pm \Delta x) = f_i \pm \Delta x\, f_i' + \frac{\Delta x^2}{2!} f_i'' \pm \frac{\Delta x^3}{3!} f_i''' + \cdots$$

Then

$$\frac{f_{i+1} - f_{i-1}}{2\Delta x} = f_i' + \frac{\Delta x^2}{6} f_i''' + \cdots$$

ODE (IVP) Methods on One Page!

For $du/dt = f(u)$: $\quad du/dt \approx (u_{n+1} - u_n)/\Delta t = f(u)$

$$u_{n+1} = u_n + \Delta t\, f(u_n) \quad \text{(Forward Euler)}$$

FE is explicit, stable if Δt is small, & first-order accurate.

$$u_{n+1} = u_n + \Delta t\, f(u_{n+1}) \quad \text{(Backward Euler)}$$

BE is implicit, A-stable & L-stable, & first-order accurate.

$$u_{n+1} = u_n + \frac{\Delta t}{2}\left(f(u_n) + f(u_{n+1})\right) \quad \text{(TR)}$$

TR is implicit, A-stable but not L-stable, & second-order accurate.

Plus One More Page: 4th Order Runge-Kutta

For $du/dt = f(u)$, RK4:

$$u_{n+1} = u_n + \frac{\Delta t}{3}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$\begin{aligned}
K_1 &= \frac{1}{2} f(u_n) \\
K_2 &= \frac{1}{2} f(u_n + \Delta t\, K_1) \\
K_3 &= \frac{1}{2} f(u_n + \Delta t\, K_2) \\
K_4 &= \frac{1}{2} f(u_n + 2\Delta t\, K_3)
\end{aligned}$$

RK4 is explicit & conditionally stable.

First-Order Wave (Advection) Equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad u(x, t = 0) = u_0(x)$$

Solution is

$$u(x, t) = u_0(x - ct)$$

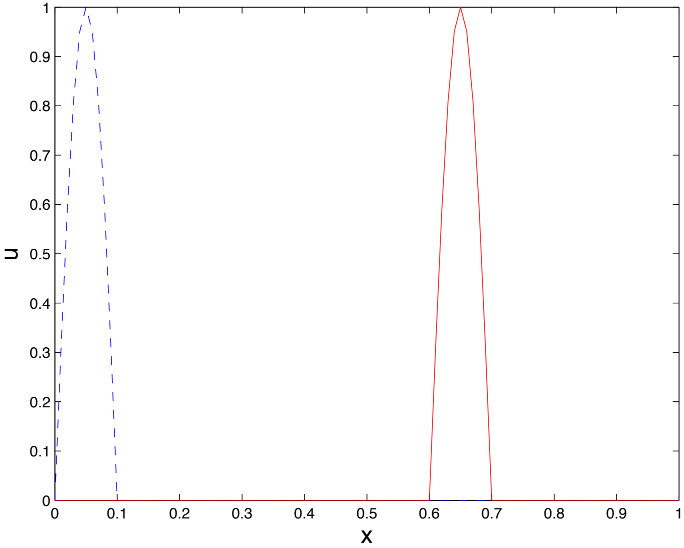
Numerical method: *upwind*

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -c\frac{u_i^n - u_{i-1}^n}{\Delta x}$$

$$u_i^{n+1} = u_i^n - c\frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right)$$

First-order accurate & stable for $\Delta t \leq \frac{\Delta x}{c}$ (CFL condition)

Wave Equation



Diffusion Equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \;\; u(x, t=0) = u_0(x), \;\; u(x=-1, t) = 0 = u(x=1, t)$$

Numerical method: *backward Euler*

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = D\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

$$u_i^{n+1} = u_i^n + D\frac{\Delta t}{\Delta x^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)$$

L-stable & first-order accurate

In matrix form with $u = [u_2, u_3, \ldots, u_N]^t$ & $u_1 = 0 = u_{N+1}$,

$$u^{n+1} = u^n + \frac{D\Delta t}{\Delta x^2}Au^{n+1}$$

$$\left(I - \frac{D\Delta t}{\Delta x^2}A\right)u^{n+1} = u^n$$

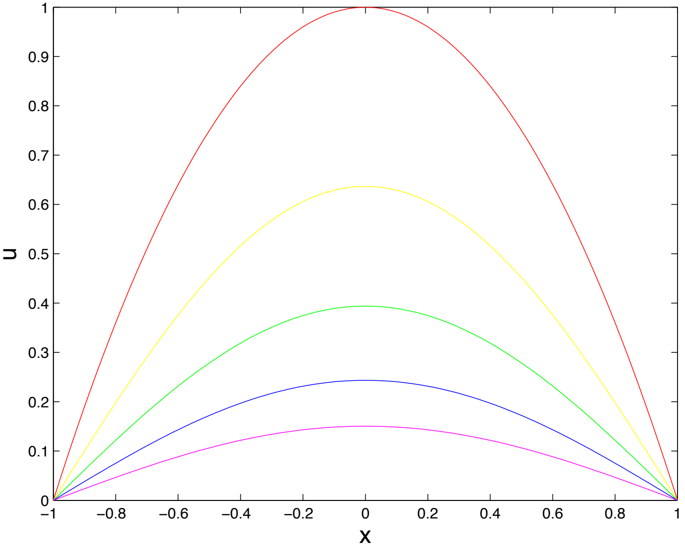
Diffusion Equation

$$\left(I - \frac{D\Delta t}{\Delta x^2} A\right) u^{n+1} = u^n$$

where (for $N = 10$)

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Diffusion Equation



Poisson's Equation & Laplace's Equation

Poisson equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\rho(x, y)$$

Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Numerical method: *second-order accurate central differences & Gauss-Seidel iteration*

Define residual (set $\Delta x = \Delta y = h$)

$$r_{ij} = \left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}\right)/h^2 \to 0$$

Jacobi iteration

$$u_{ij}^{(k+1)} = \frac{1}{4}\left(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k)}\right) = u_{ij}^{(k)} + \frac{h^2}{4} r_{ij}^{(k)}$$

Laplace’s Equation