Electrical & Reaction-Diffusion Modeling of Biological Cells

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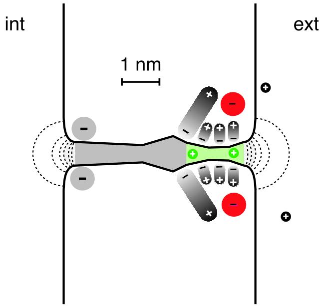
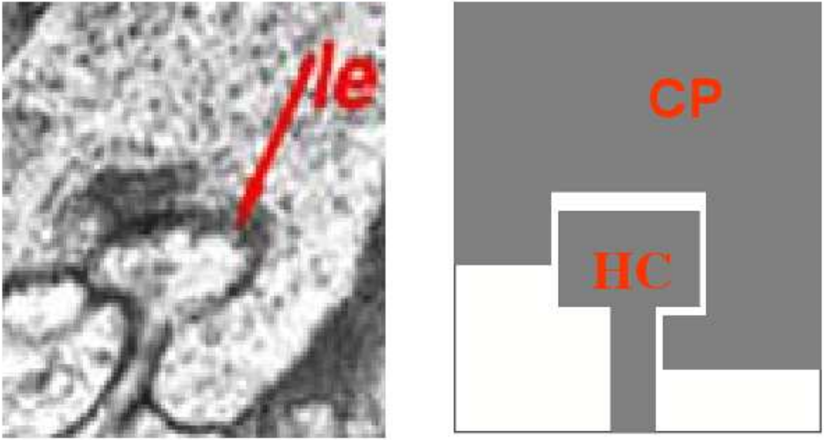
(a) Numerical Methods for Drift-Diffusion Model

Schematic view of K channel structure





Electron micrograph & schematic of the horizontal cell dendrite contacting a cone pedicle (1 micron by 1 micron)

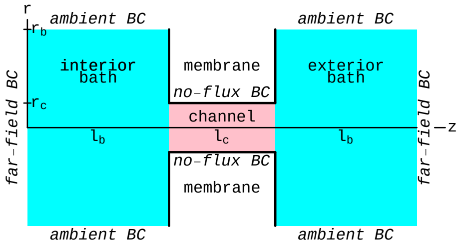
Drift-Diffusion Model

$i = K^{+}, Cl^{-}, Na^{+}, Ca^{++}, \ldots$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot \left(q_i \mu_i \mathbf{E} n_i \right) = D_i \nabla^2 n_i$$

$$\nabla \cdot \left(\epsilon \nabla \phi \right) = eN - \sum_i q_i n_i, \quad \mathbf{E} = -\nabla \phi$$

parabolic/elliptic system of PDEs



$$n_i = n_{bi}, \quad \phi = 0 \ \ \text{(interior bath far-field BC)}$$

$$n_i = n_{bi}, \quad \phi = V \ \ \text{(exterior bath far-field BC)}$$

$$n_i = n_{bi}, \quad \frac{\partial \phi}{\partial r} = 0 \ \ \text{(ambient bath BC)}$$

$$\hat{\mathbf{n}} \cdot \nabla n_i = 0, \quad \hat{\mathbf{n}} \cdot \nabla \phi = 0 \ \ \text{(no-flux BC)}$$

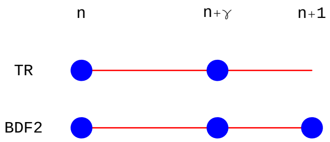
TRBDF2 Numerical Method

$$\frac{du}{dt} = f(u,t), \quad \gamma = 2 - \sqrt{2}$$

$$u^{n+\gamma} - \gamma \frac{\Delta t_n}{2} f^{n+\gamma} = u^n + \gamma \frac{\Delta t_n}{2} f^n \quad (\mathbf{TR})$$

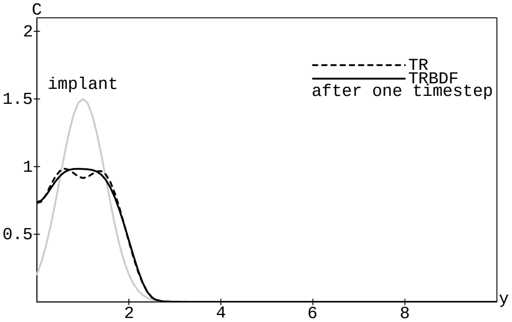
$$u^{n+1} - \frac{1-\gamma}{2-\gamma} \Delta t_n f^{n+1} = \frac{1}{\gamma(2-\gamma)} u^{n+\gamma} - \frac{(1-\gamma)^2}{\gamma(2-\gamma)} u^n \quad (\mathbf{BDF2})$$

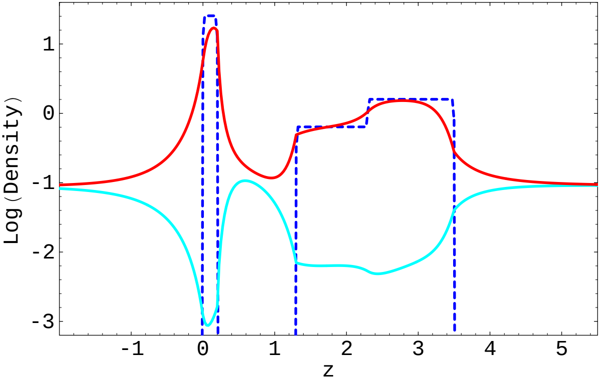
Use Newton's method if $f(u)$ is nonlinear



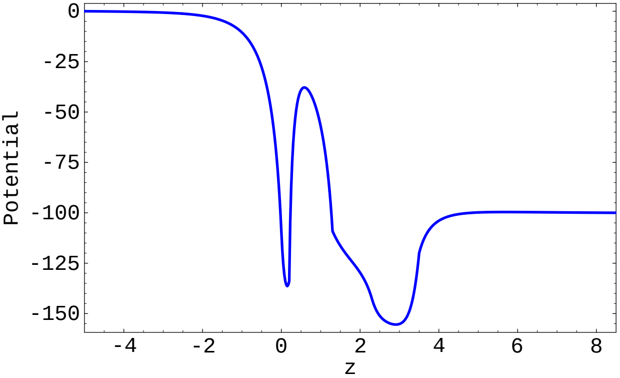
Advantages of TRBDF2

- One-step (composite) method
- Second-order accurate & L-stable
- Easy to adjust Δt dynamically

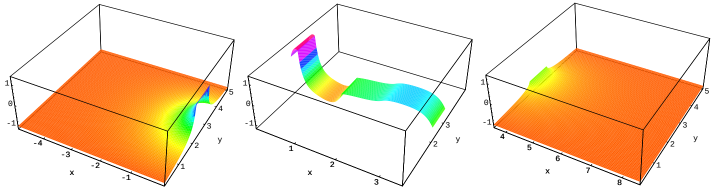




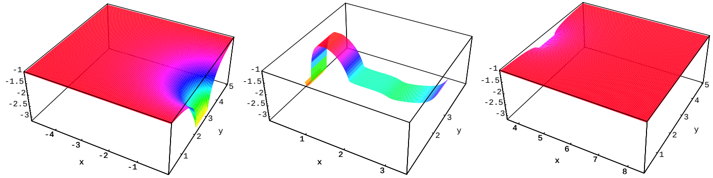
K^+ (red), Cl^- (cyan), & $-$ve fixed charge (blue)
for $V = -100$ mV



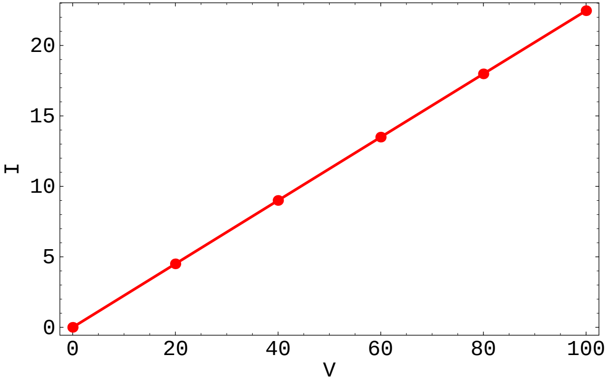
Electrostatic potential in mV for $V = -100$ mV



Log(K^+) for $V = -100$ mV

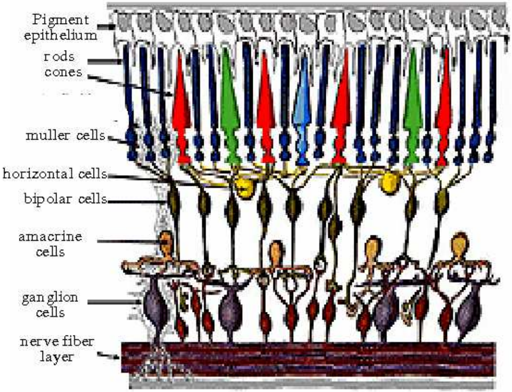


Log(Cl^-) for $V = -100$ mV



Current in picoamperes vs. voltage in mV *(b) Modeling Nerve Cells in the Retina*

with Steve Baer, Shaojie Chang, Sharon Crook, & Chris Ringhofer



Continuum Model for a 2D Horizontal Cell Sheet

$$\tau_m \frac{\partial V_c}{\partial t} = -(V_c - V_{Lc}) - (V_c - V_{Cl})\, G_{Cl} R_m + I_{app} R_m$$

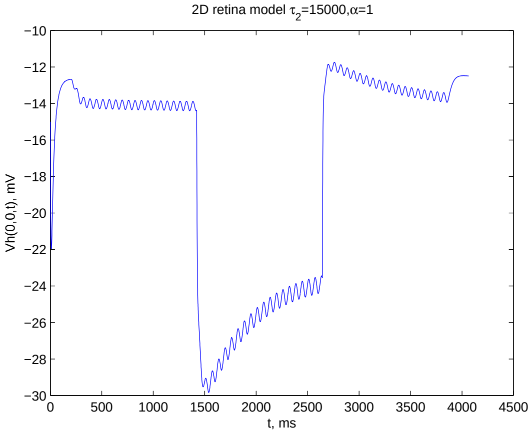
$$\tau_m \frac{\partial V_h}{\partial t} = \lambda^2 \left(\frac{\partial^2 V_h}{\partial x^2} + \frac{\partial^2 V_h}{\partial y^2} \right) - (V_h - V_{Lh}) + \overline{n} R_S \frac{(U_h - V_h)}{R_{ss}} + I_{ext} R_m$$

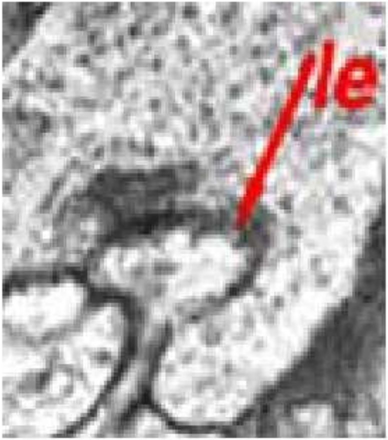
$$\tau_m \frac{\partial U_h}{\partial t} = -(U_h - V_{Lh}) - \frac{R_m}{A_{sh}} \frac{(U_h - V_h)}{R_{ss}} - k_6 R_m [GL] U_h$$

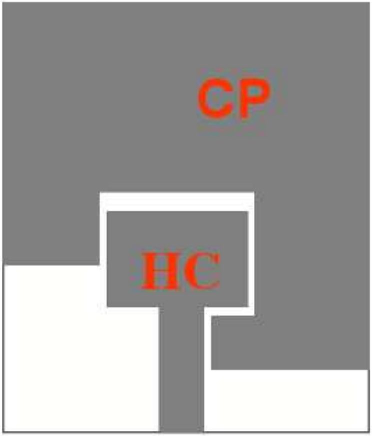
$$\tau_2 \frac{\partial [S]}{\partial t} = k_2 \left(U_h - \frac{58 \ln([S]/[S_I])}{n_i \ln 10} \right)$$

$$\tau_0 \frac{\partial [Ca]}{\partial t} = \frac{A}{2} \left(\frac{1 + \tanh\left(a(V_c - \alpha U_h + 20)\right)}{k_{Oca}[S] + 1} \right) - [Ca]$$

$$\tau_5 \frac{\partial [GL]}{\partial t} = k_4 [Ca] - k_5 [GL] (E_{Na} - U_h)$$


2D retina model τ_2=15000,α=1
Vh(0,0,t), mV
t, ms
−10
−12
−14
−16
−18
−20
−22
−24
−26
−28
−30
0
500
1000
1500
2000
2500
3000
3500
4000
4500





Simulate 400 nm by 400 nm micron region with gap ≈ 20–40 nm

A Model of the Membrane