Tensor Factorization and Spin construction for Kac-Moody algebras

Rajeev Walia

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Factorization Results

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Thus, $W = 2^r\,\mathrm{Spin}_0(\mathfrak{g}^{\perp})$

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Factorization Results

Factorization formula II

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$\tilde{\mu} =$ any dominant weight of $\widetilde{\mathfrak{g}}$.
If

$$V(\tilde{\mu}) \downarrow^{\widetilde{\mathfrak{g}}}_{\mathfrak{g}} \cong \bigoplus_{i=1}^{k} V(\mu_i)$$

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$\tilde{\mu} =$ any dominant weight of $\widetilde{\mathfrak{g}}$.

If

$$V(\tilde{\mu}) \downarrow_{\mathfrak{g}}^{\widetilde{\mathfrak{g}}} \;\cong\; \bigoplus_{i=1}^{k} V(\mu_i)$$

Then,

$$V(2\tilde{\mu} + \tilde{\rho}) \downarrow_{\mathfrak{g}}^{\widetilde{\mathfrak{g}}}$$

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Then,

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$\widetilde{\mathfrak{g}} = \mathfrak{sl}_n\mathbb{C}$.

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$$X \mapsto \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 0 & \text{n-1} \\ & & & & & 0 \end{pmatrix}$$

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$$Y \mapsto \begin{pmatrix} 0 & & & & & \\ \text{n-1} & 0 & & & & \\ & \text{n-2} & 0 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 1 & 0 \end{pmatrix}$$

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$H \mapsto \text{diag}[\text{n-1} , \text{n-3} , \ldots , -(\text{n-3}) , -(\text{n-1})]$.

$S_\lambda(x_1, \ldots, x_n)$

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is called the "Principal specialization" of a Schur polynomial.

Combinatorial consequences of Factorization Formulae

Consequence 1 :Principal specialization(contd.)

$$S_{2\mu+\rho}(1, q, q^2, \cdots, q^{n-1})$$

$$S_{2\mu+\rho}(1, q, q^2, \cdots, q^{n-1})$$
$$= \left(q^{\binom{n}{3}}(1+q)\, S_{\mu}(1, q^2, q^4, \ldots, q^{2n-2}\,)\right) \cdot (\; w_1(q) \cdot w_2(q) \cdots w_{n-2}(q)\;),$$

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where $w_k(q) = (1+q)(1+q^2) \cdots (1+q^{k+1})$.

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where $w_k(q) = (1+q)(1+q^2)\cdots(1+q^{k+1})$.

The non-trivial fact is that all $(n-1)$ factors on the right-hand side of the formula are *symmetric unimodal* q-polynomials.

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This is a kind of a multiplicative analog of a result of Reiner and Stanton

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The non-trivial fact is that all $(n-1)$ factors on the right-hand side of the formula are *symmetric unimodal* q-polynomials.

This is a kind of a multiplicative analog of a result of Reiner and Stanton which states that for certain pairs λ, μ, the centered difference $S_\lambda(1, \ldots, q^{n-1}) - q^N S_\mu(1, \ldots, q^{n-1})$ is symmetric unimodal.

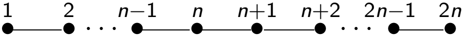
$\widetilde{\mathfrak{g}} = \mathfrak{sl}_{2n+1}\mathbb{C}$

$$\widetilde{\mathfrak{g}} = \mathfrak{sl}_{2n+1}\mathbb{C} = \ A_{2n}$$

Combinatorial consequences of Factorization Formulae

Consequence 2 : Folding of Dynkin diagrams

$\widetilde{\mathfrak{g}} = \mathfrak{sl}_{2n+1}\mathbb{C} = \ A_{2n} \ :$



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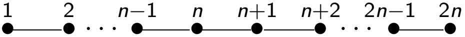

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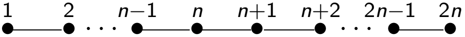

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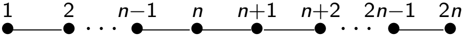

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Combinatorial consequences of Factorization Formulae

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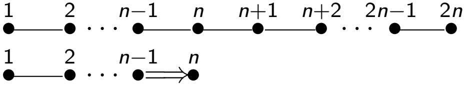

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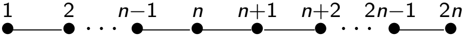
$\widetilde{\mathfrak{g}} = \mathfrak{sl}_{2n+1}\mathbb{C} = \ A_{2n}\ :$ 

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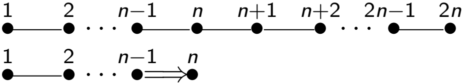

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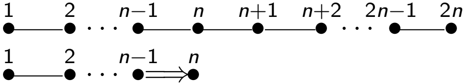

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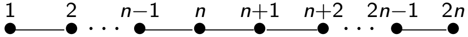
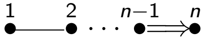
$\tilde{\mathfrak{g}} = \mathfrak{sl}_{2n+1}\mathbb{C} = \ A_{2n} \ :$ 

$\mathfrak{g} = \mathfrak{so}_{2n+1}\mathbb{C} = \ B_n \ :$ 

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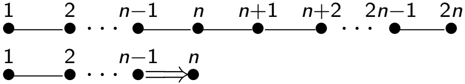

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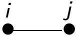
$e =$ the number of edges 

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$$\Lambda = \frac{1}{2} \sum_{\beta \in S^+} m_\beta \beta$$

if V is finite dimensional.
In general,

$$\Lambda = \sum_{i=1}^{n} c_i \Lambda_i$$

where the coefficient $c_i \geq 0$ is defined as follows:

Character formula for Spin (contd.)

Character formula for Spin (contd.)

Let $a := \alpha_i(d)$ and for $(j,k) \in \mathbb{Z} \times \mathbb{Z}$, let

$$b_{j,k} := \sum_{\substack{\beta(\alpha_i^\vee)=j \\ \beta(d)=k}} m_\beta \ .$$

Then we have:

$$c_i := \sum_{\substack{j\,,\,k \\ 0<k<aj}} \tfrac{1}{2}\left(b_{j,k} - b_{j+2,k+a}\right)\left(j - \left\lfloor \tfrac{k}{a} \right\rfloor\right)\left(1 + \left\lfloor \tfrac{k}{a} \right\rfloor\right) \ .$$

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a $\widetilde{\mathfrak{g}}$-representation called the affinization of V.

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where $\theta^\vee$ is the coroot of the highest root of $\mathfrak{g}$.

Affine Lie algebras

Refinement of earlier Tensor factorization results for Affinized representations

$$(1) \qquad V(\hat{\rho})\downarrow^{\hat{\mathfrak{g}}}_{\mathfrak{g}}$$

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where $U := 2^{m_0+n} \, V(2\mu+\rho) \otimes V(2\mu+\rho)$
if $\mathrm{Spin}_0(V) \cong V(\mu)$ and $\mathrm{Spin}_0(\hat{V}) \cong V(\hat{\mu})$

Future Goals

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- Reconstruct the structure of $\widetilde{\mathfrak{g}}$-representation on tensor product.
- Apply Spin on Kirilov Reshetikhin modules.
- Explore full category $\mathcal{I}_O$ versus $\mathcal{I}_A$.
- Extend Spin_0 to "virtual representations".

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