Physical Combinatorics

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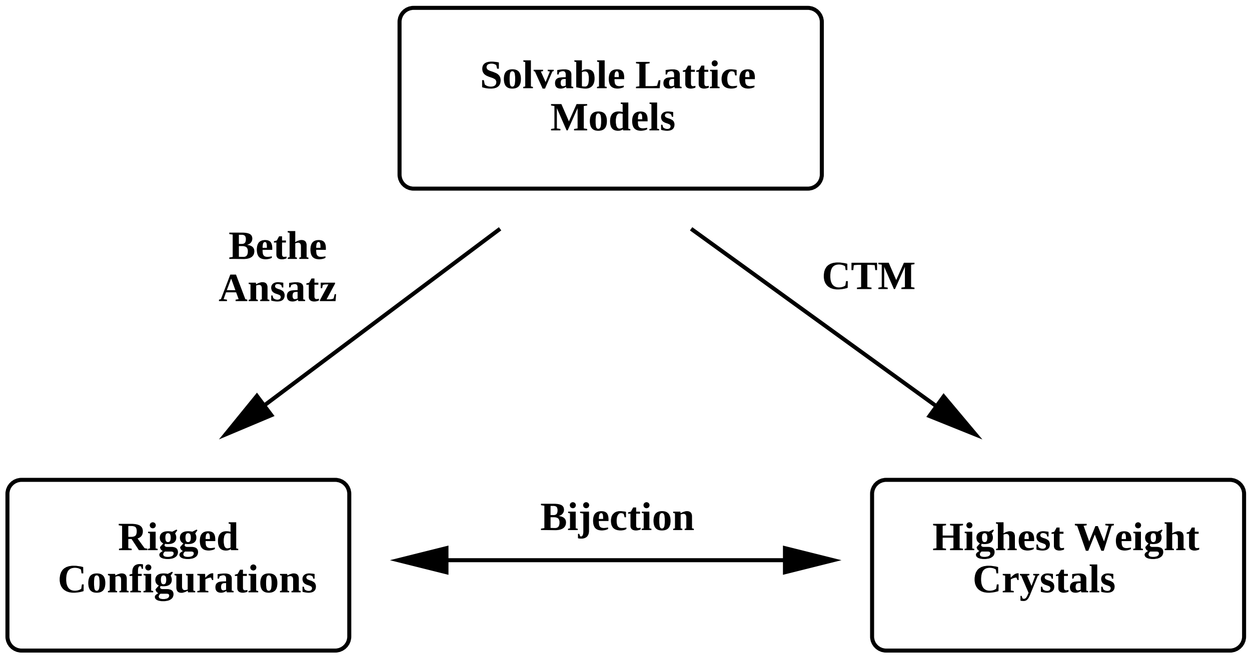
AMS meeting, Tuscon
April 22, 2007

Motivation



1988 Identity for Kostka polynomials Kerov, Kirillov, Reshetikhin

2001 $X = M$ conjecture of HKOTTY

Outline

1. Rogers-Ramanujan identities, fractional statistics, and the $X = M$ conjecture
2. Kirillov-Reshetikhin crystals

Rogers-Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})}$$

where $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ for $n > 0$ and $(q)_0 = 1$.

Some History

- proved in a paper by Rogers in 1894
- conjectured by Ramanujan in a letter to Hardy in 1913;
  published in 1916 in the book *Combinatory Analysis* by MacMahon without proof
- new proof in 1917 by Rogers and Ramanujan
- different independent proof by Schur in 1917

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Rogers-Schur-Ramanujan identities

Partition interpretation

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}$$

$S = \{s_1, s_2, s_3, \ldots\}$ set

$$\prod_{n \in S} \frac{1}{1-q^n} = \prod_{n \in S} (1 + q^n + q^{2n} + q^{3n} + \cdots)$$
$$= (1 + q^{s_1} + q^{2s_1} + q^{3s_1} + \cdots)$$
$$\times (1 + q^{s_2} + q^{2s_2} + q^{3s_2} + \cdots) \cdots .$$

Partition interpretation

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$S = \{s_1, s_2, s_3, \ldots\}$ set

$$\prod_{n\in S} \frac{1}{1-q^n} = \prod_{n\in S}(1+q^n+q^{2n}+q^{3n}+\cdots)$$
$$=(1+q^{s_1}+q^{2s_1}+q^{3s_1}+\cdots)$$
$$\times(1+q^{s_2}+q^{2s_2}+q^{3s_2}+\cdots)\cdots.$$

Theorem. The product side is the generating function of partitions with parts congruent 1 or 4 modulo 5.

Example: The coefficient of q^6 is 3 since there are three partitions of 6 with parts congruent to 1 or 4 modulo 5:

$$(1,1,1,1,1,1), \quad (4,1,1) \quad \text{and} \quad (6).$$

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Is there an interpretation of the sum side of the RR identities?

Some more history

- debut of the Rogers–Ramanujan identities in physics made by Baxter in 1981 in a paper on the Hard Hexagon model
- in 1990's the Stony Brook group interpreted the Rogers–Ramanujan identities as the partition function of a physical system with quasiparticles obeying certain exclusion statistics
  $\Rightarrow$ fermionic formulas
- HKOTTY in 1999/2001 conjectured fermionic formulas for all Kac–Moody Lie algebras
  $\Rightarrow$ $X = M$ conjecture

The Hard Hexagon model

Set of paths:

height variable $\sigma_i \in \{0, 1\}$ for $0 \leq i \leq L$

boundary condition $\sigma_0 = \sigma_L = 0$

requirement $\sigma_i \sigma_{i+1} = 0$

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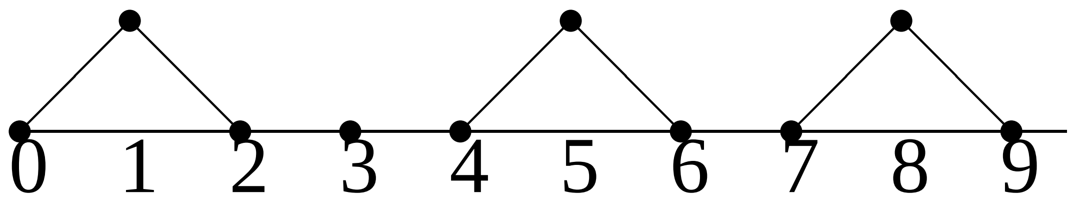
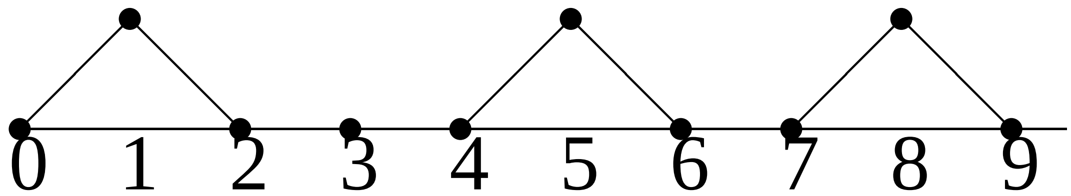
Example: Path of length 9

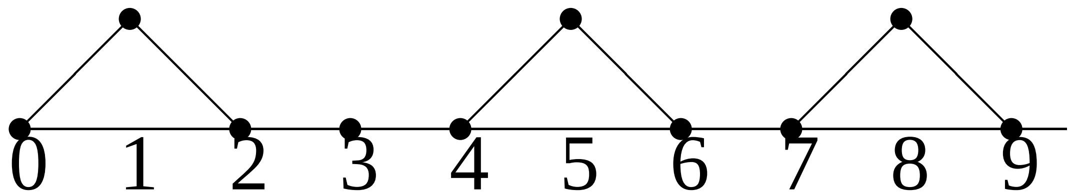




Energy function

$$E(p) = \sum_{j=1}^{L} j\sigma_j$$

$$E(p) = 1 + 5 + 8 = 14$$



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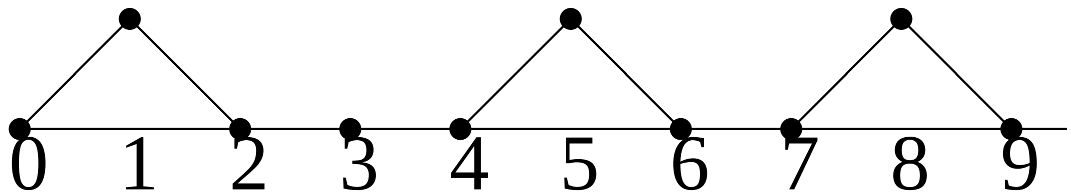

Energy function

$$E(p) = \sum_{j=1}^{L} j\sigma_j$$

Generating function

$$X(L) = \sum_{p \text{ path of length } L} q^{E(p)}$$

Explicit formula

Recurrence: $X(L)$ is completely determined by $X(0) = X(1) = 1$ and

$$X(L) = X(L-1) + q^{L-1}X(L-2).$$

Theorem. $X(L) = \sum_{n=0}^{\infty} q^{n^2} \begin{bmatrix} L-n \\ n \end{bmatrix} =: M(L)$

Corollary. $\lim_{L\to\infty} M(L) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}$

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Sum side of the RR identities

Partition interpretation

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}$$

Theorem. The sum side is the generating function of partitions for which the difference between any two parts is at least two.

Partition interpretation

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}$$

Theorem. The sum side is the generating function of partitions for which the difference between any two parts is at least two.

Example. Partitions of 6 with the difference between any two parts at least two are

$$(4,2), \quad (5,1) \quad \text{and} \quad (6).$$

Statistics

Bosons: adding a particle does not remove any states from the system

$$\sum_{m=0}^{\infty} \frac{q^m}{(q)_m} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

generating function of all partitions

Fermions: adding a particle removes one state from the system

$$\sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} = \prod_{n=0}^{\infty} (1+q^n)$$

generating function of partitions with distinct parts

Fractional statistics

RR identity: interpret each triangle in a path as a particle; adding a particle removes two states from the system

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}$$

Fractional statistics

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Haldane statistics:

d_a: dimension of Hilbert space for particles of type a

N_a: number of particles of type a

g_{ab}: statistics matrix

$$\Delta d_a = -\sum_b g_{ab} \Delta N_b$$

Marriage

Citation from Dyson's famous paper "Missed opportunities" (1972)

> "As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce... I shall examine in detail some examples of missed opportunities, occasions on which mathematicians and physicists lost chances of making discoveries by neglecting to talk to each other."

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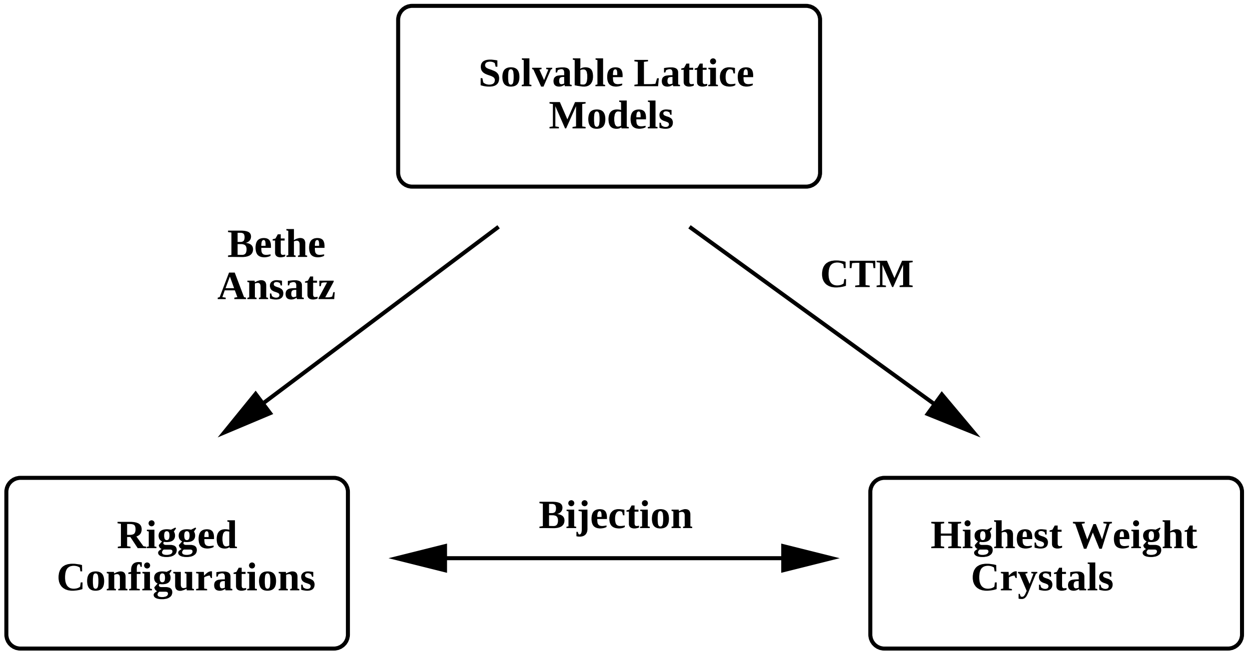

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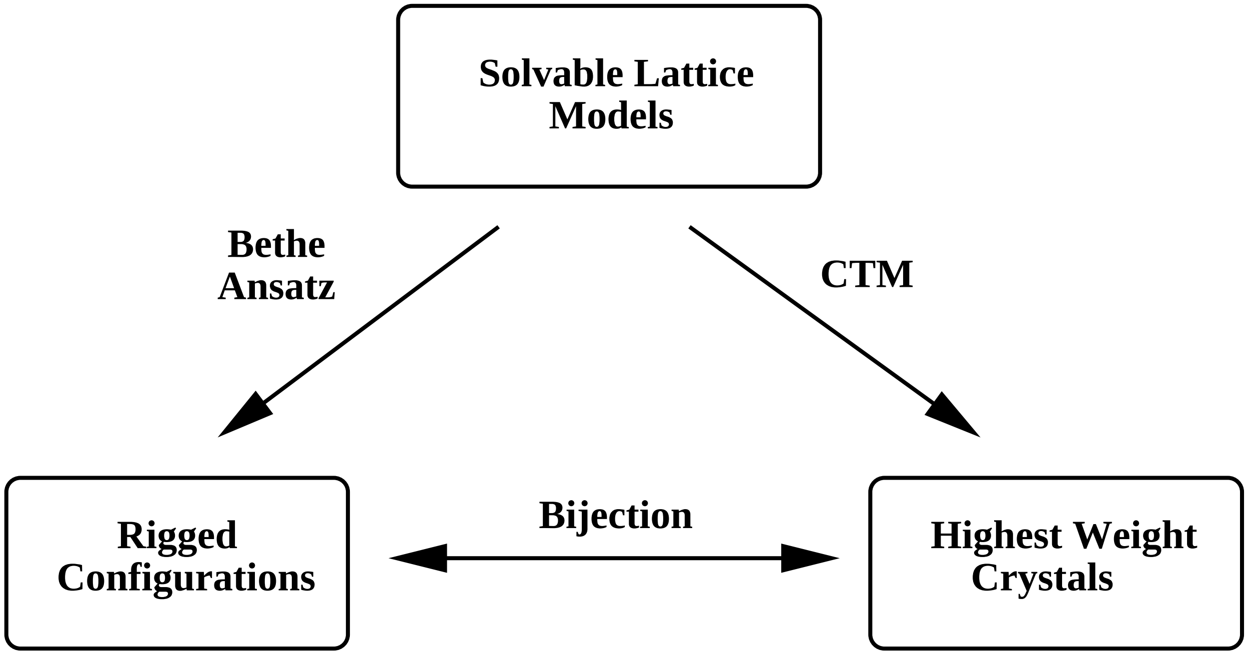


1988 Identity for Kostka polynomials Kerov, Kirillov, Reshetikhin

2001 $X = M$ conjecture of HKOTTY

$\rightsquigarrow$ Kirillov–Reshetikhin (KR) crystals

References

This talk is based on the following papers:

- A. Schilling,
  Combinatorial structure of Kirillov–Reshetikhin crystals of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$,
  preprint math.QA/0704.2046
- M. Okado, A. Schilling,
  Uniqueness of Kirillov–Reshetikhin crystals,
  in preparation

Outline

Combinatorial structure of KR crystals of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$

- Crystals
- KR crystals
- Dynkin diagram automorphisms
- Classical crystal structure
- Affine crystal structure
- MuPAD-Combinat implementation
- Outlook and open problems

Quantum algebras

Drinfeld and Jimbo $\sim$ 1984:
independently introduced quantum groups $U_q(\mathfrak{g})$

Kashiwara $\sim$ 1990:
crystal bases, bases for $U_q(\mathfrak{g})$-modules as $q \longrightarrow 0$
combinatorial approach

Lusztig $\sim$ 1990:
canonical bases
geometric approach

Applications in...

representation theory
$\rightsquigarrow$ tensor product decomposition
solvable lattice models
$\rightsquigarrow$ one point functions
conformal field theory
$\rightsquigarrow$ characters
number theory
$\rightsquigarrow$ modular forms
Bethe Ansatz
$\rightsquigarrow$ fermionic formulas
combinatorics
$\rightsquigarrow$ tableaux combinatorics
topological invariant theory
$\rightsquigarrow$ knots and links

Crystals

$\mathfrak{g}$ symmetrizable Kac-Moody algebra
P weight lattice of $\mathfrak{g}$
I index of the Dynkin diagram
$\{\alpha_i \in P \mid i \in I\}$ simple roots
$\{h_i \in P^* = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \mid i \in I\}$ simple coroots

Crystals

A $U_q(\mathfrak{g})$-crystal is a nonempty set B with maps

$$\mathrm{wt}\colon B \to P$$

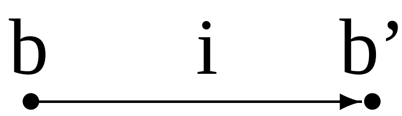
$$e_i, f_i\colon B \to B \cup \{\emptyset\} \quad \text{for all } i \in I$$

satisfying

$$f_i(b) = b' \Leftrightarrow e_i(b') = b \qquad \text{if } b, b' \in B$$

$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \qquad \text{if } f_i(b) \in B$$

$$\langle h_i\, ,\, \mathrm{wt}(b) \rangle = \varphi_i(b) - \epsilon_i(b)$$

Write 

for $b' = f_i(b)$

KR crystals

$\mathfrak{g}$ affine Kac–Moody algebra

$W^{r,s}$ KR module indexed by $r \in \{1, \ldots, n\}$, $s \geq 1$

$\rightsquigarrow$ finite-dimensional $U'_q(\mathfrak{g})$-module

Chari proved

$$W^{r,s} \cong \bigoplus_{\Lambda} W(\Lambda) \quad \text{as } U_q(\mathfrak{g}_0)\text{-module}$$

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$\mathfrak{g}$ of type $A_n^{(1)} \Rightarrow \mathfrak{g}_0$ of type A_n

$$W^{r,s} \cong W\left(\underbrace{\left.\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}\right\} r}_{s} \right)$$

KR crystals

$\mathfrak{g}$ affine Kac–Moody algebra

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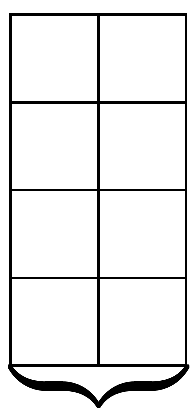
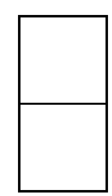
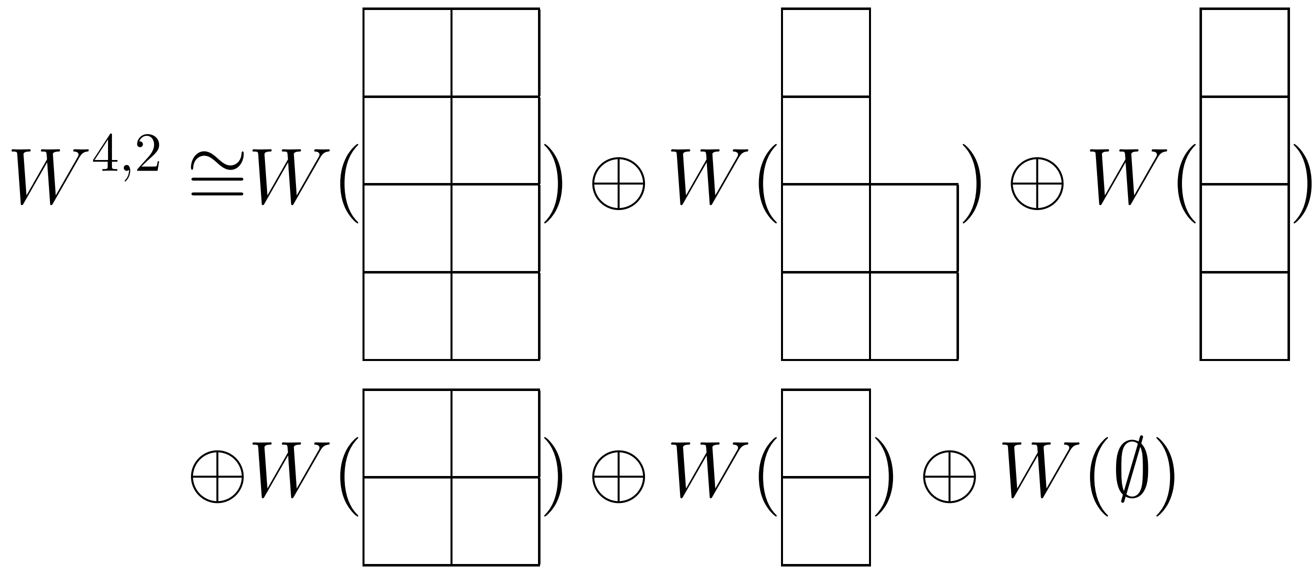
$\mathfrak{g}$ of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$ $\Rightarrow$ $\mathfrak{g}_0$ of type D_n, B_n, C_n

sum over  r with vertical dominos  removed

s

Example

Type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$

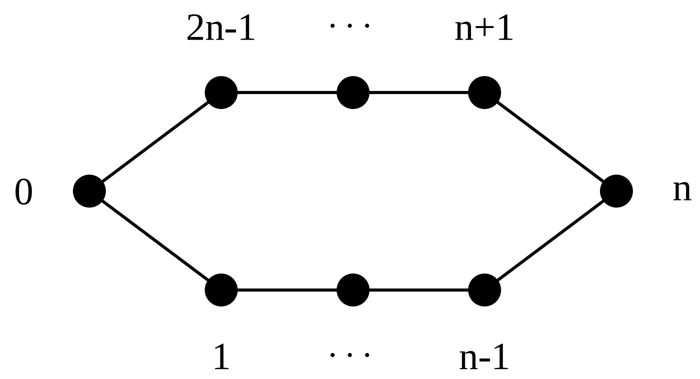


Dynkin automorphism

Type $A_n^{(1)}$:

KKMMNN proved existence of crystals $B^{r,s}$ for $W^{r,s}$
Shimozono gave the combinatorial structure of $B^{r,s}$
using σ

$A_{2n-1}^{(1)}$



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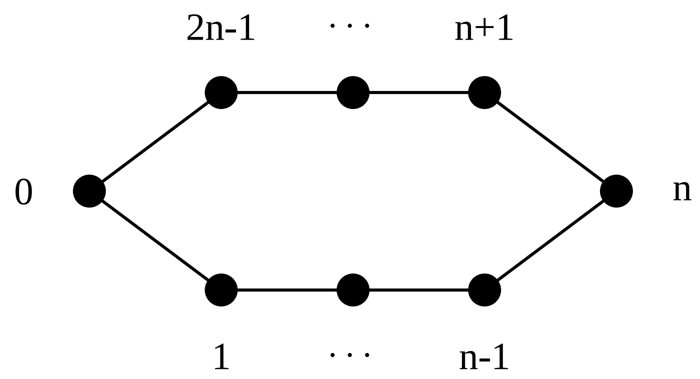
$A^{(1)}_{2n-1}$



$$e_0 = \sigma^{-1} \circ e_1 \circ \sigma$$

$$f_0 = \sigma^{-1} \circ f_1 \circ \sigma$$

Dynkin automorphism

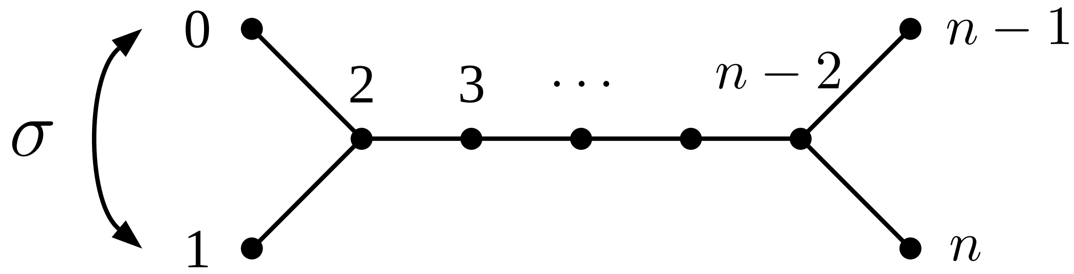
Type $D_n^{(1)}$:

Okado proved existence of crystals $B^{r,s}$ for $W^{r,s}$
S., Sternberg combinatorial structure of $B^{2,s}$
Sternberg conjecture for $B^{r,s}$

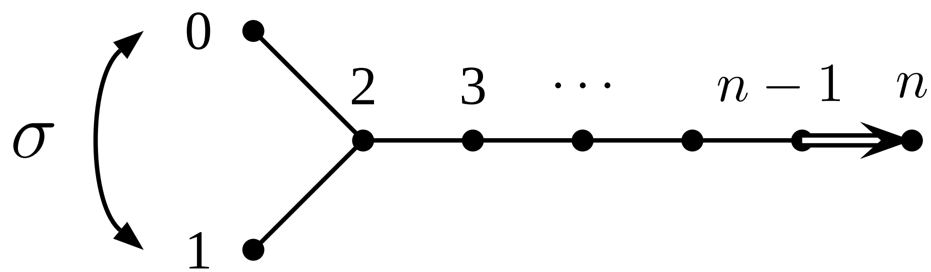
Here we give the combinatorial structure of $B^{r,s}$ for type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$ using the Dynkin automorphism σ

Dynkin automorphism

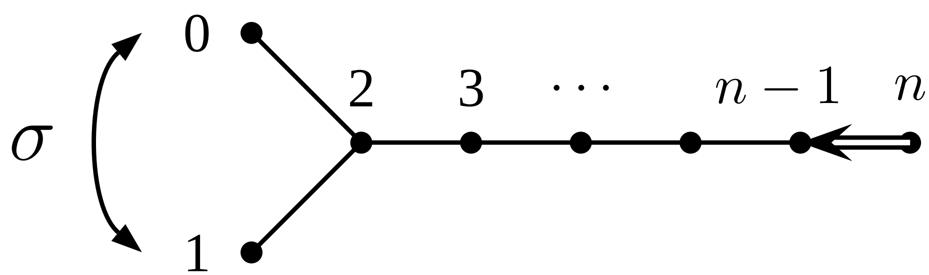
Type $D_n^{(1)}$:



Type $B_n^{(1)}$:

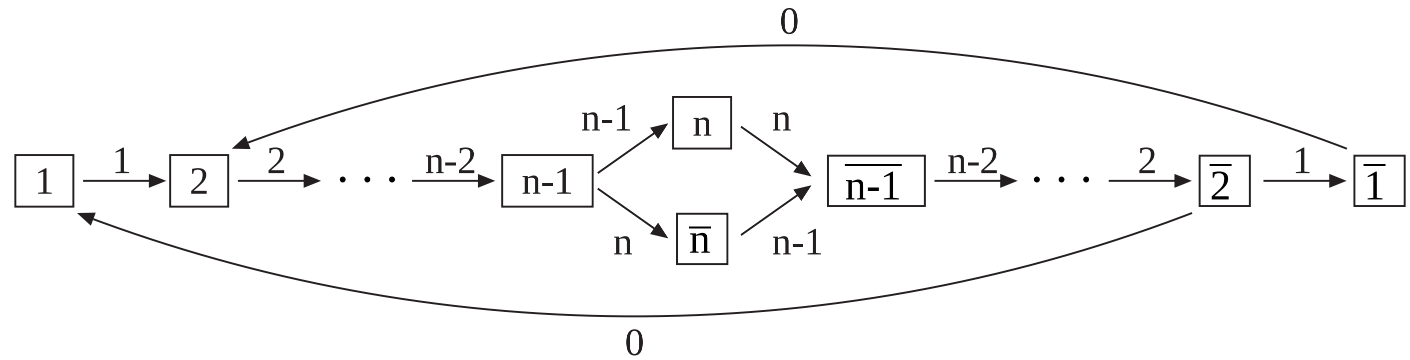
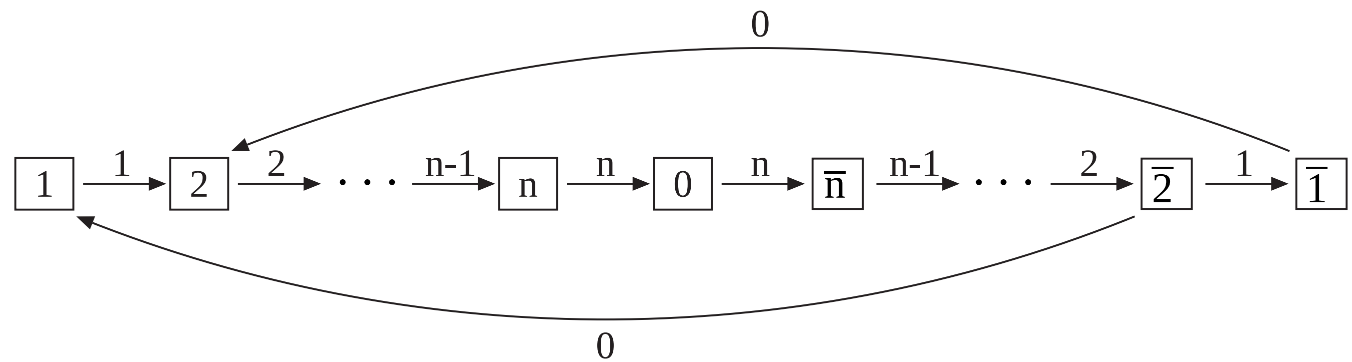
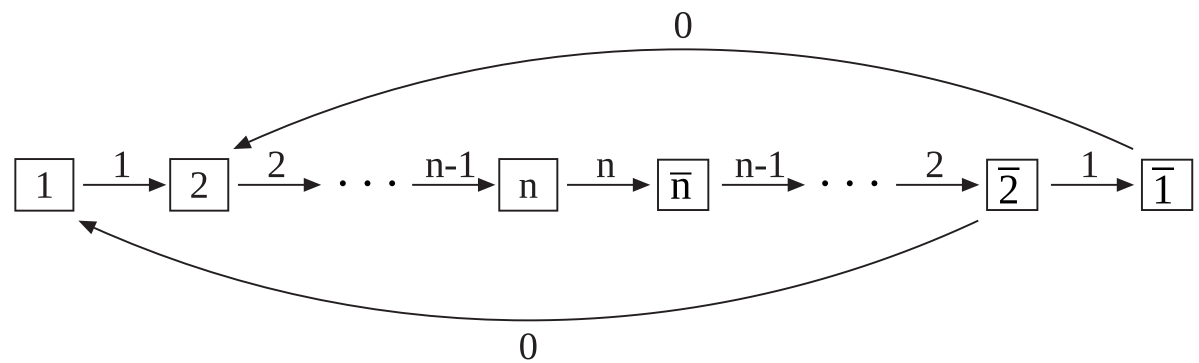


Type $A_{2n-1}^{(1)}$:



$$e_0 = \sigma \circ e_1 \circ \sigma \quad \text{and} \quad f_0 = \sigma \circ f_1 \circ \sigma$$

Crystals $B^{1,1}$

$D_n^{(1)}$	
$B_n^{(1)}$	
$A_{2n-1}^{(2)}$	

Classical decomposition

By construction

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$

as $X_n = D_n, B_n, C_n$ crystals

$\Rightarrow$ crystal arrows f_i, e_i are fixed for $i = 1, 2, \ldots, n$

Classical crystal

$$B(\lambda) \subset B(\boxed{})^{\otimes |\lambda|}$$

highest weight

4		
3		
2	2	2
1	1	1

$$\mapsto \boxed{4} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{1}$$

f_i, e_i for $i = 1, 2, \ldots, n$ act by tensor product rule

$$b \quad \otimes \quad b'$$

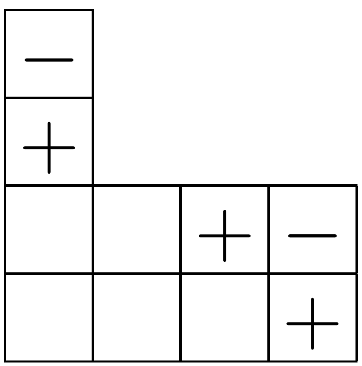
$$\underbrace{- \; - \; -}_{\varphi_i(b)}\underbrace{+ \; + \; +}_{\varepsilon_i(b)} \qquad \underbrace{- \; -}_{\varphi_i(b')}\underbrace{+ \; + \; + \; +}_{\varepsilon_i(b')}$$

Definition of σ

$D_n \to D_{n-1}$ branching

$$B_{D_n}(\Lambda) \cong \bigoplus_{\substack{\pm \text{ diagrams } P \\ \text{outer}(P) = \Lambda}} B_{D_{n-1}}(\text{inner}(P))$$

$\pm$ diagrams



$\lambda \subset \mu \subset \Lambda$

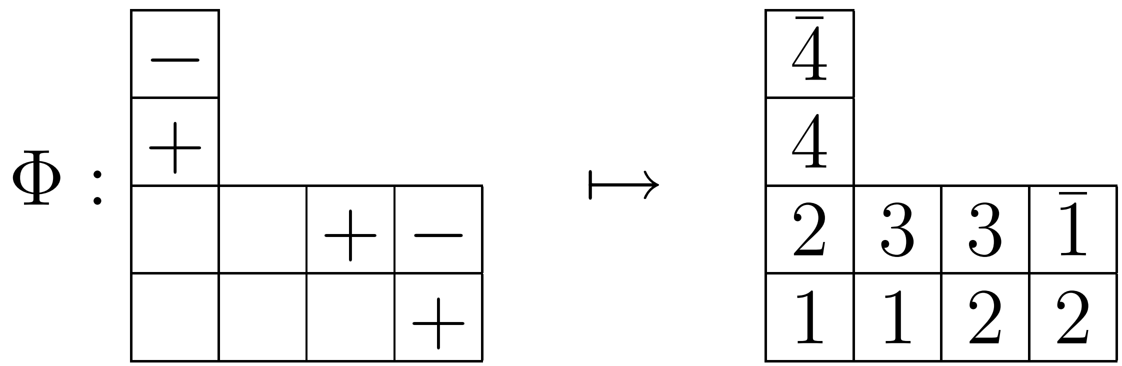
inner shape outer shape

Λ/μ horizontal strip filled with $-$

μ/λ horizontal strip filled with $+$

Definition of σ

D_{n-1} highest weight vectors
are in bijection with $\pm$ diagrams via Φ



Definition of σ

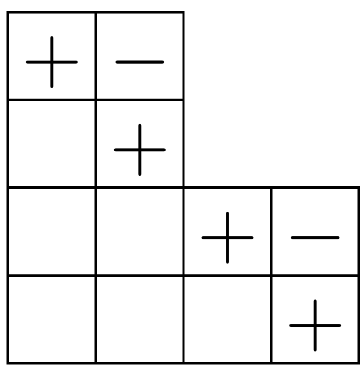
σ on $\pm$ diagrams

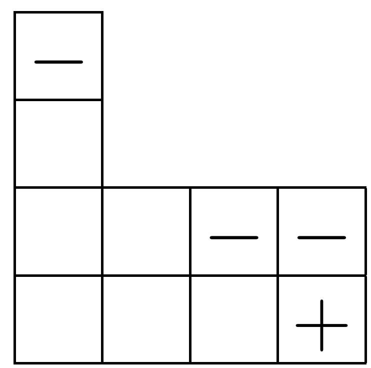
P $\pm$ diagram of shape Λ/λ

columns of height h in λ

$$h \equiv r-1 \bmod 2: \quad \text{interchange number of } + \text{ and } - \text{ above } \lambda$$

$$h \equiv r-1 \bmod 2: \quad \text{interchange number of } \mp \text{ and empty above } \lambda$$

$P =$ 

$\mathfrak{S}(P) =$ 

$r \geq 6$

$s = 5$

Definition of σ

σ on tableaux

$b \in B^{r,s}$

$$e_{\vec{\mathbf{a}}} := e_{a_1} \cdots e_{a_\ell} \quad \text{such that } e_{\vec{\mathbf{a}}}(b) \text{ is } D_{n-1} \text{ highest weight vector}$$

$$f_{\overleftarrow{\mathbf{a}}} := f_{a_\ell} \cdots f_{a_1}$$

Then

$$\sigma(b) = f_{\overleftarrow{\mathbf{a}}} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\vec{\mathbf{a}}}(b)$$

Example

$B^{4,5}$ of type $D_6^{(1)}$

$$b = \begin{array}{|c|c|cc} \hline \bar{4} & \bar{4} & & \\ \hline 3 & 4 & & \\ \hline 2 & 3 & \multicolumn{1}{c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \xrightarrow{e_4 e_6 e_5 e_4 e_3 e_2 e_2} \begin{array}{|c|c|cc} \hline 4 & \bar{4} & & \\ \hline 3 & 4 & & \\ \hline 2 & 3 & \multicolumn{1}{c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

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$$\xrightarrow{\Phi^{-1}} \begin{array}{|c|c|cc}\hline + & - & & \\ \hline & + & & \\ \hline & & \multicolumn{1}{|c|}{-} & \multicolumn{1}{c|}{-} \\ \hline & & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{+} \\ \hline \end{array} \xrightarrow{\mathfrak{S}} \begin{array}{|c|ccc}\hline - & & & \\ \hline & & & \\ \hline & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{+} & \multicolumn{1}{c|}{-} \\ \hline & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{+} \\ \hline \end{array}$$

Example

$B^{4,5}$ of type $D_6^{(1)}$

$$b = \begin{array}{|c|c|cc}\hline \bar{4} & \bar{4} & & \\ \hline 3 & 4 & & \\ \hline 2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \xrightarrow{e_4 e_6 e_5 e_4 e_3 e_2 e_2} \begin{array}{|c|c|cc}\hline 4 & \bar{4} & & \\ \hline 3 & 4 & & \\ \hline 2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

$$\xrightarrow{\Phi^{-1}} \begin{array}{|c|c|cc}\hline + & - & & \\ \hline & + & & \\ \hline & & \multicolumn{1}{|c|}{-} & \multicolumn{1}{c|}{-} \\ \hline & & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{+} \\ \hline \end{array} \xrightarrow{\mathfrak{S}} \begin{array}{|c|ccc}\hline - & & & \\ \hline & & & \\ \hline & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{+} & \multicolumn{1}{c|}{-} \\ \hline & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{+} \\ \hline \end{array}$$

$$\xrightarrow{\Phi} \begin{array}{|c|ccc}\hline \bar{3} & & & \\ \hline 4 & & & \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \xrightarrow{f_2 f_2 f_3 f_4 f_5 f_6 f_4} \begin{array}{|c|ccc}\hline \bar{2} & & & \\ \hline \bar{4} & & & \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} = \sigma(b)$$

Sketch of Proof

Theorem[FSS]

The KR crystals $B^{r,s}$ of type $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$ are uniquely determined by the following properties:

1. As an X_n crystal, $B^{r,s}$ decomposes according as
$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \quad \text{where } X_n = D_n, B_n, C_n.$$
2. $B^{r,s}$ is regular.
3. There is a unique element $u \in B^{r,s}$ such that
$$\varepsilon(u) = s\Lambda_0 \quad \text{and} \quad \varphi(u) = \begin{cases} s\Lambda_0 & \text{for } r \text{ even,} \\ s\Lambda_1 & \text{for } r \text{ odd.} \end{cases}$$
4. $B^{r,s}$ admits the automorphism σ.

Sketch of Proof

Theorem[FSS]

The KR crystals $B^{r,s}$ of type $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$ are uniquely determined by the following properties:

...

Proof via embedding of Demazure crystal into $B^{r,s}$
$\Rightarrow$ completely fixes 0-arrows

Sketch of Proof

Condition 1: Classical decomposition holds by construction.
Condition 4: Existence of σ holds by construction.
Condition 3: Existence of u for r even

$$u = \emptyset \in B(\emptyset)$$

$$\Rightarrow \mathfrak{S} \circ \Phi^{-1}(u) = \underbrace{\begin{array}{|c|c|c|c|c|c|}\hline - & - & - & - & - & - \\ \hline + & + & + & + & + & + \\ \hline\end{array}}_{s}$$

$$\Rightarrow \tilde{u} = \Phi \circ \mathfrak{S} \circ \Phi^{-1}(u) = \begin{array}{|c|c|c|c|c|c|}\hline \bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{1} \\ \hline 1 & 1 & 1 & 2 & 2 & 2 \\ \hline\end{array}$$

$$\varepsilon(\tilde{u}) = s\Lambda_1 \quad \varphi(\tilde{u}) = s\Lambda_1$$

$$\varepsilon(u) = s\Lambda_0 \quad \varphi(u) = s\Lambda_0$$

Sketch of Proof

Condition 1: Classical decomposition holds by construction.
Condition 4: Existence of σ holds by construction.
Condition 3: Existence of u for r odd

$$u = \underbrace{\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}}_{s} \in B(s\omega_1)$$

$$\Rightarrow \mathfrak{S} \circ \Phi^{-1}(u) = \underbrace{\boxed{-}\boxed{-}\boxed{-}\boxed{-}\boxed{-}\boxed{-}}_{s}$$

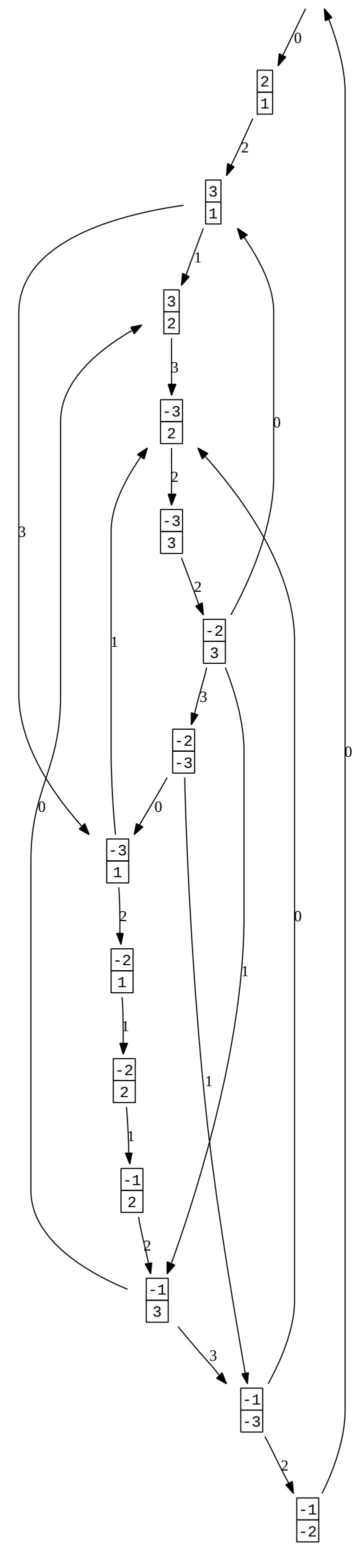
$$\Rightarrow \tilde{u} = \Phi \circ \mathfrak{S} \circ \Phi^{-1}(u) = \boxed{\bar{1}}\boxed{\bar{1}}\boxed{\bar{1}}\boxed{\bar{1}}\boxed{\bar{1}}\boxed{\bar{1}}$$

$$\varepsilon_1(\tilde{u}) = s \quad \varphi_1(\tilde{u}) = 0$$

$$\varepsilon(u) = s\Lambda_0 \quad \varphi(u) = s\Lambda_1$$

Example

$B^{2,1}$ type $A_5^{(2)}$



Sketch of Proof

Condition 2: Regularity of crystal

Need to show: for every $K \subset I = \{0, 1, \ldots, n\}$ with $|K| = 2$ the K-component of $B^{r,s}$ is the corresponding $U_q(\mathfrak{g}_K)$-crystal

Sketch of Proof

Condition 2: Regularity of crystal

Need to show: for every $K \subset I = \{0, 1, \ldots, n\}$ with $|K| = 2$ the K-component of $B^{r,s}$ is the corresponding $U_q(\mathfrak{g}_K)$-crystal

$K = \{i, j\}, i, j \neq 0$ clear by construction

Sketch of Proof

Condition 2: Regularity of crystal
Need to show: for every $K \subset I = \{0, 1, \dots, n\}$ with $|K| = 2$ the K-component of $B^{r,s}$ is the corresponding $U_q(\mathfrak{g}_K)$-crystal

$K = \{0, i\}, i \neq 1$

$$e_0 e_i = \sigma e_1 \sigma e_i = \sigma(e_1 \sigma e_i \sigma)\sigma = \sigma(e_1 e_i)\sigma$$

Sketch of Proof

Condition 2: Regularity of crystal

Need to show: for every $K \subset I = \{0, 1, \ldots, n\}$ with $|K| = 2$ the K-component of $B^{r,s}$ is the corresponding $U_q(\mathfrak{g}_K)$-crystal

$K = \{0, 1\}$ need to show $e_0 e_1 = e_1 e_0$

hard part!!

MuPAD-Combinat...

... is an open source algebraic combinatorics package for the computer algebra system MuPAD

```
>> KR:=crystals::kirillovReshetikhin(2,2,["D",4,1]):
>> t:=KR([[3],[1]])

                                        +---+
                                        | 3 |
                                        +---+
                                        | 1 |
                                        +---+
>> t::e(0)

                                        +----+
                                        | -2 |
                                        +----+
                                        | 3  |
                                        +----+
```

MuPAD-Combinat...

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```
>> KR:=crystals::kirillovReshetikhin(2,2,["D",4,1]):
>> t:=KR([[3],[1]])

                                        +---+
                                        | 3 |
                                        +---+
                                        | 1 |
                                        +---+

>> t::sigma()

                                     +----+----+
                                     | -2 | -1 |
                                     +----+----+
                                     | 2  | 3  |
                                     +----+----+
```

Open Problems

- Existence and combinatorial structure for other KR crystals $C_n^{(1)}$, $D_{n+1}^{(2)}$,...
- Characterization of unrestricted rigged configurations (done for type $A_n^{(1)}$)
- Fermionic formulas for unrestricted Kostka polynomials
  Relation to fermionic formulas of [HKKOTY]?
- Relation to other rigged configurations [S]
  $\rightsquigarrow$ LLT polynomials
- Relation to box ball systems, description in terms of R-matrices
- Level restriction