An approach through combinatorics to intersection number questions for the moduli spaces of curves

David M. Rhŷs Jackson

University of Waterloo

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This is joint research at various times with: George Andrews, Ian Goulden, John Harer, Malcolm Perry, Alex Vainshtein, Ravi Vakil, Terry Visentin

Outline

I Maps in orientable and non-orientable surfaces

- Basic facts about embeddings, and examples
- An algebraic encoding of maps

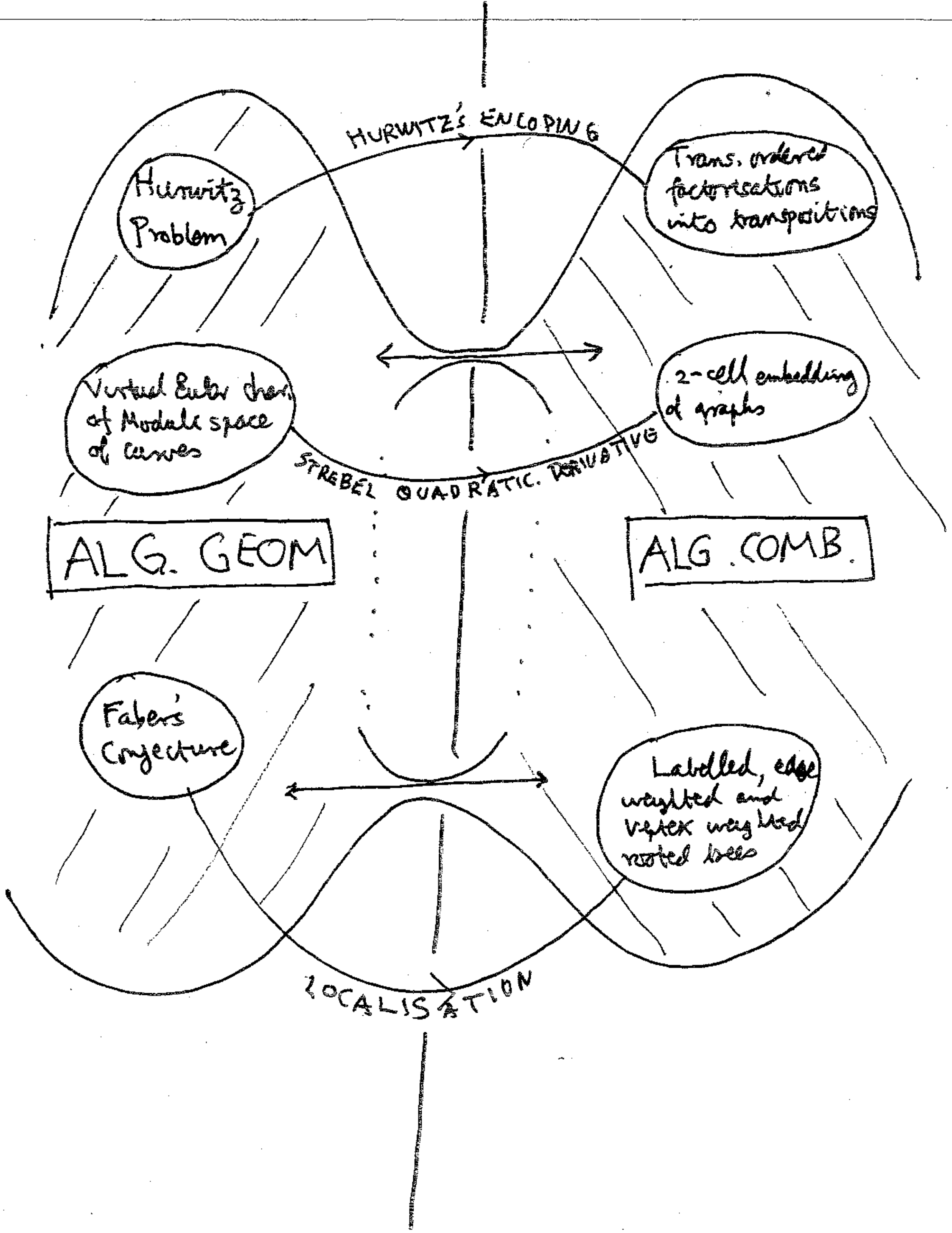
II The map series and two examples

- Jack symmetric functions and an integral representation
- Two models from mathematical physics
- Aspects of the moduli spaces of algebraic curves

III Maps and the moduli space of smooth curves

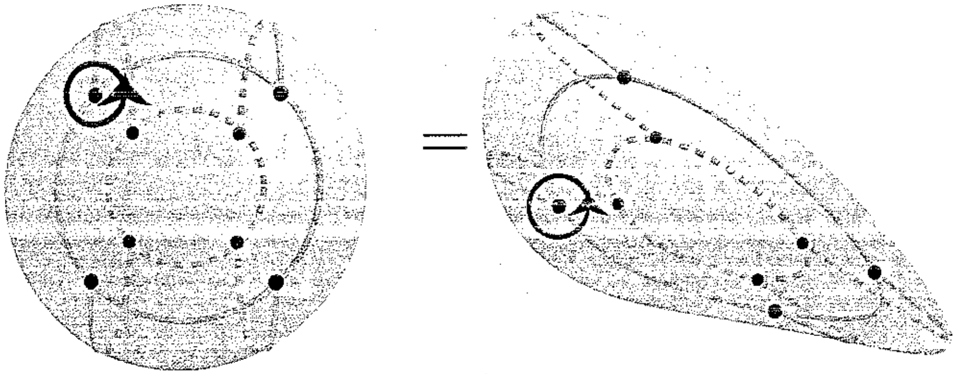
- Faber's Top Intersection Number Conjecture
- An approach for all genera through localisation and algebraic combinatorics

(1.1)

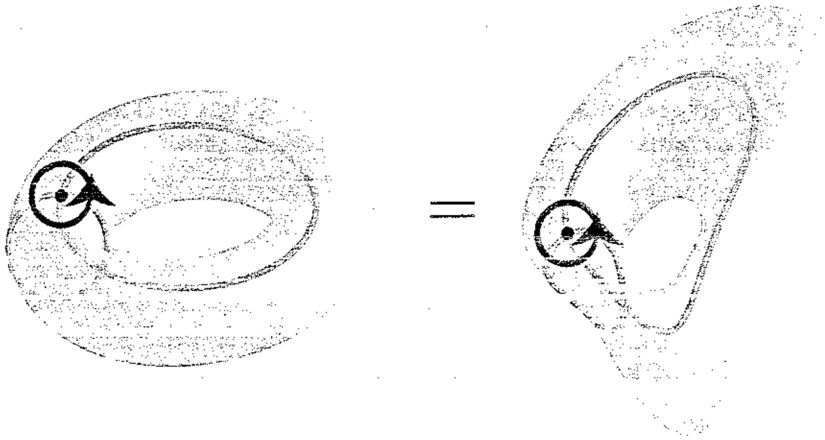

HURWITZ'S ENCODING
Hurwitz Problem
Trans. ordered factorisations into transpositions
Virtual Euler char. of Moduli space of curves
2-cell embedding of graphs
STREBEL QUADRATIC DERIVATIVE
ALG. GEOM
ALG. COMB.
Faber's Conjecture
Labelled, edge weighted and vertex weighted rooted trees
LOCALISATION

Part I

Maps in orientable and non-orientable surfaces

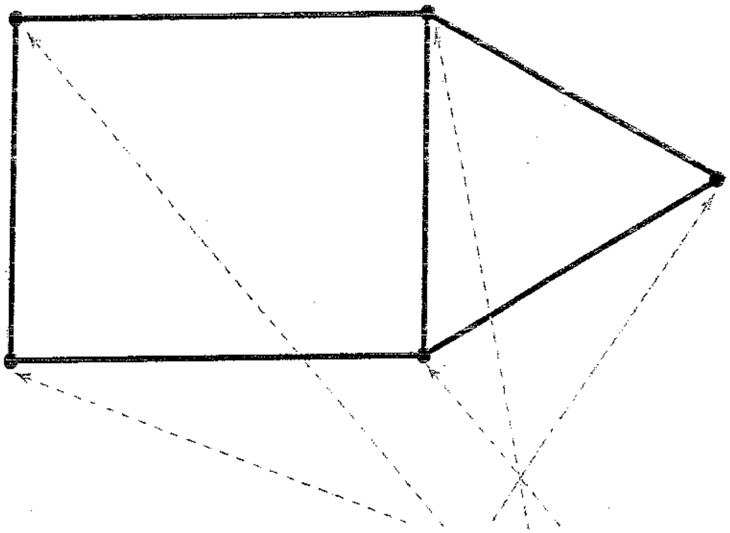


Two maps are said to be equivalent if one can be obtained from the other by smooth deformations of the surface

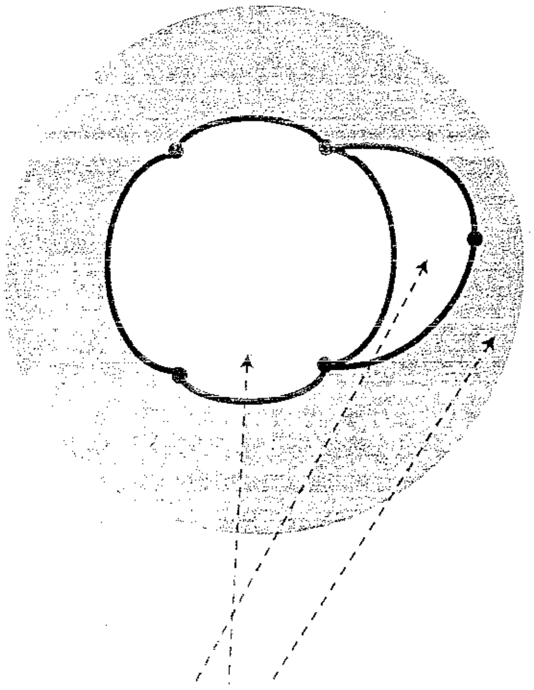


The cyclic order of edges around a vertex is invariant under smooth deformations of the surface

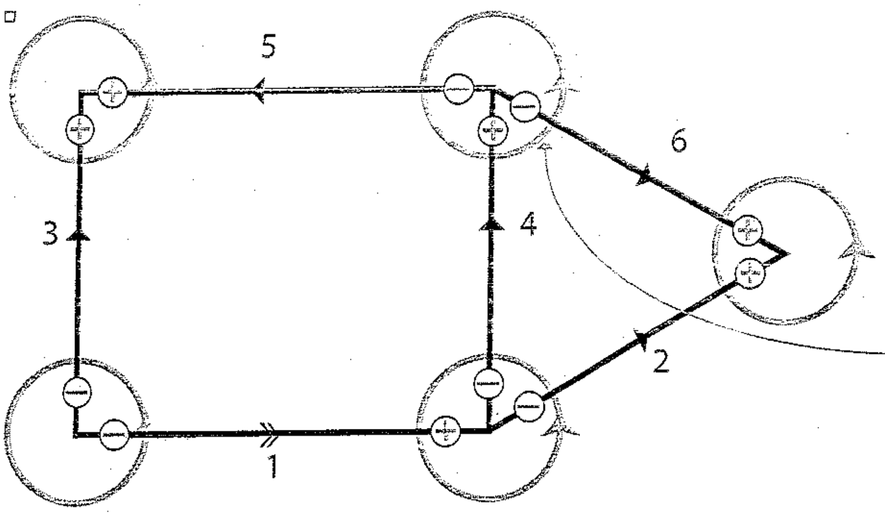
I.2 - The Embedding Theorem



Encode vertex degrees **by** [2 2 2 3 3]



Encode face degrees **by** [3 4 5]



An anticlockwise circulation around each vertex can be consistently defined

For example, the anticlockwise circulation around this vertex is

$$(4^+\,6^-\,5^-)$$

- Let $\nu = (5^+\,3^+)\cdot(1^-\,3^-)\cdot(1^+\,2^-\,4^-)\cdot(2^+\,6^+)\cdot(4^+\,6^-\,5^-) \in \mathfrak{S}_{12}$, the vertex permutation for the map
- Let $\varepsilon = (1^+\,1^-)\cdot\cdots\cdot(6^+\,6^-) \in \mathfrak{S}_{12}$, the edge permutation of the map

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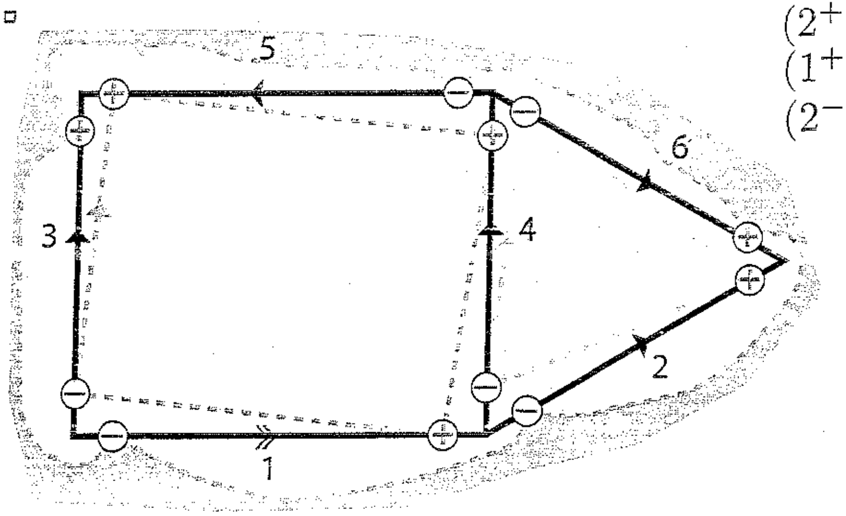
Theorem (Embedding). *The vertex permutation ν uniquely defines a map. The faces of the map correspond to the cycles of the face permutation $\nu\varepsilon$*

Demonstration of the use of the Embedding Theorem



$(2^+\, 4^-\, 6^-)$: 1 of degree 3
$(1^+\, 3^-\, 5^+\, 4^+)$: 1 of degree 4
$(2^-\, 6^+\, 5^-\, 3^+\, 1^-)$: 1 of degree 5

Face-type is [3 4 5]

3 vertices of degree 2
2 vertices of degree 3
Vertex-type is [2 2 2 3 3]

- Let $\nu = (5^+\, 3^+) \cdot (1^-\, 3^-) \cdot (1^+\, 2^-\, 4^-) \cdot (2^+\, 6^+) \cdot (4^+\, 6^-\, 5^-) \in \mathfrak{S}_{12}$, the vertex permutation for the map: cycle-type $[2^3\, 3^2] \vdash 12$. This is also the vertex-type of the map.

- Then $\phi := \nu\varepsilon = (1^+\, 3^-\, 5^+\, 4^+) \cdot (2^+\, 4^-\, 6^-) \cdot (2^-\, 6^+\, 5^-\, 3^+\, 1^-) \in \mathfrak{S}_{12}$ is the face permutation of the map: cycle-type $[3\, 4\, 5] \vdash 12$

Part II

The map series and two applications

II.1 - The maps series

Definition. *The map series is* the formal sum

$$M_{\mathcal{A}}^{O}(u,x,y,z) = \sum_{\mathfrak{m}\in\mathfrak{M}_{\mathcal{A}}^{O}} u^{g(\mathfrak{m})} x^{\#_v(\mathfrak{m})} y^{\#_f(\mathfrak{m})} z^{\#_e(\mathfrak{m})} \in \mathbb{C}[u,x,y]\,[[z]]$$

where

- $\mathfrak{M}_{\mathcal{A}}^{O}$ *is the set of all maps in orientable surfaces with face degrees in the set* $\mathcal{A}$
- $\#_v(\mathfrak{m})$ *is the number of vertices of the map* $\mathfrak{m}$ *etc.;* $g(\mathfrak{m})$ *is the genus of* $\mathfrak{m}$

- The coefficient $[u^g x^i y^j z^k]M_{\mathcal{A}}^{O}$ of $u^g x^i y^j z^k$ in $M_{\mathcal{A}}^{O}$ is the number of maps in $\mathfrak{M}_{\mathcal{A}}^{O}$ with • genus g, • i vertices, • j faces, • k edges

Theorem. *Let $\Pi_{\mathcal{A}}$ be the set of all partitions with parts in $\mathcal{A}$, and let $\mathcal{C}_\alpha$ be the conjugacy class of $\mathfrak{S}_{2n}$ indexed by $\alpha \vdash 2n$.*

$$M_{\mathcal{A}}^{O}(u^2, x, y, z) = 2u^2 z \frac{\partial}{\partial z} \log R_{\mathcal{A}}^{O}(xu^{-1}, yu^{-1}, uz/2)$$

where

$$R_{\mathcal{A}}^{O}(x, y, z) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\substack{\nu, \phi \vdash 2n, \\ \phi \in \Pi_{\mathcal{A}}}} |\mathcal{C}_\nu \varepsilon \cap \mathcal{C}_\phi| \, x^{l(\nu)} y^{l(\phi)}$$

- Embedding Theorem - vertex partition ν
- Remove decoration - labelling and orienting of edges
- Retain only connected maps - this device was known to Hurwitz
- Euler's formula - $\#_v(\mathfrak{m}) - \#_e(\mathfrak{m}) + \#_f(\mathfrak{m}) = 2 - 2g(\mathfrak{m})$

- $\mathbb{C}\mathfrak{S}_{2n}$ denotes the group algebra of $\mathfrak{S}_{2n}$ over $\mathbb{C}$
- For $\alpha \vdash 2n$, let
$$\mathsf{K}_\alpha = \sum_{\pi \in \mathcal{C}_\alpha} \pi$$
- The centre $\mathcal{Z}_{\mathbb{C}\mathfrak{S}_{2n}}$ of $\mathbb{C}\mathfrak{S}_{2n}$ is spanned by K_α for $\alpha \vdash 2n$,
- Let $\nu, \phi \vdash 2n$. Then
$$|\mathcal{C}_\nu \varepsilon \cap \mathcal{C}_\phi| = \frac{|\mathcal{C}_\phi|}{|\mathcal{C}_{[2^n]}|}[\mathsf{K}_\phi]\,\mathsf{K}_\nu \mathsf{K}_{[2^n]}$$
- $\mathcal{Z}_{\mathbb{C}\mathfrak{S}_{2n}}$ has a basis consisting of orthogonal idempotents
- The orthogonal idempotents are linear combinations of the K_α with scalars that are evaluations of characters χ^θ of irreducible representations of $\mathfrak{S}_{2n}$.
- We use the orthogonal idempotents to determine $[\mathsf{K}_\phi]\,\mathsf{K}_\nu \mathsf{K}_{[2^n]}$ and thence the combinatorial number $|\mathcal{C}_\nu \varepsilon \cap \mathcal{C}_\phi|$

Theorem. *The map series for orientable surfaces* is

$$M_{\mathcal{A}}^{O}(u^2, x, y, z) = 2u^2 z \frac{\partial}{\partial z} \log R_{\mathcal{A}}^{O}(xu^{-1}, yu^{-1}, uz/2)$$

where

$$R_{\mathcal{A}}^{O}(x, y, z) = \sum_{n \geq 0} \frac{z^n}{n!(2n)!} \sum_{\phi \in \Pi_{\mathcal{A}}} |\mathcal{C}_\phi| y^{l(\phi)} \sum_{\theta \vdash 2n} \chi_\phi^\theta \chi_{[2^n]}^\theta H_\theta(x)$$

and

- χ_ϕ^θ *is the evaluation of the character* χ^θ *on the conjugacy class* $\mathcal{C}_\phi$
- $H_\theta(x) = \prod_{1 \leq i \leq l(\theta)} (x - i + 1)^{(\theta_i)}$ where $\theta = (\theta_1, \theta_2, \ldots)$, and
  $(x)^{(k)} = x(x+1) \cdots (x + k - 1)$
- $\Pi_{\mathcal{A}}$ *is the set of all partitions with parts in* $\mathcal{A}$

II.2 - Quadrangulations and partition functions

Corollary. *Let M_4^O be the map series for quadrangulations in orientable surfaces. Let M be the map series of all maps in orientable surfaces. Then*

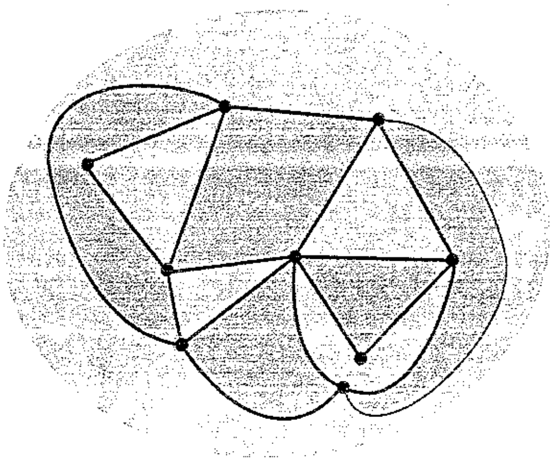
$$M_4^O(u^2, x, y, z) = \frac{1}{2}(M^O(4u^2, x+u, x, yz^2) + M^O(4u^2, x-u, x, yz^2))$$

- The proof is character theoretic
- No constructive proof is known
- A similar result holds for triangulations

- Two brief comments connected with string theory
  - M_4^O and M^O are partition functions associated with quantum chromodynamics and 2-d quantum gravity, before the scaling limit is taken
  - The former involves quark-antiquark pairs as time evolves, taking account of the interaction of gluons

II.3 - Extension to hypermaps in all surfaces

The algebraic theory is more symmetrical for hypermaps



A hypermap with:

- Vertex-type: $[2^2\,2^6\,6] \vdash 34$
- Hyperedge-type: $[3^3\,4^2] \vdash 17$
- Face-type: $[3^3\,4^2] \vdash 17$

- Vertex-type $[2^2\,4^6\,6] \vdash 34$ is encoded algebraically as $x_1^2 x_4^6 x_6^1$
- Hyperedge-type $[3^3\,4^2] \vdash 17$ is encoded as $z_3^3 z_4^2$
- Face-type $[3^3\,4^2] \vdash 17$ is encoded as $y_3^3 y_4^2$
- The corresponding group is the Double Coset Algebra $\mathfrak{H}_{2n} \diagdown \mathfrak{S}_{4n} \diagup \mathfrak{H}_{2n}$ of the hyperoctahedral group $\mathfrak{H}_{2n}$

- $p_k = x_1^k + x_2^k + \cdots$ and $p_{[a,b,\dots]} = p_a p_b \cdots$ (power sums)
- Jack symmetric functions $J_\alpha(\mathrm{x}, a)$ are orthogonal with respect to

$$\langle p_\alpha, p_\beta \rangle_a = \frac{|\alpha|!}{|\mathcal{C}_\alpha|} a^{l(\alpha)} \delta_{\alpha,\beta}$$

Theorem. *Let* $\Psi \equiv \Psi(\mathrm{p}(\mathrm{x}), \mathrm{p}(\mathrm{y}), \mathrm{p}(\mathrm{z}); b)$ *where*

$$\Psi = (1+b)t\frac{\partial}{\partial t} \log \sum_\theta \frac{t^{|\theta|}}{\|J_\theta\|_{1+b}} J_\theta(\mathrm{x}; 1+b) J_\theta(\mathrm{y}; 1+b) J_\theta(\mathrm{z}; 1+b)\Big|_{t=1}$$

Then

$$\begin{aligned} H^O(\mathrm{x},\mathrm{y},\mathrm{z}) &= \Psi(\mathrm{x},\mathrm{y},\mathrm{z};0), && \textit{hypermap series for orientable surfaces,} \\ H(\mathrm{x},\mathrm{y},\mathrm{z}) &= \Psi(\mathrm{x},\mathrm{y},\mathrm{z};1), && \textit{hypermap series for all surfaces.} \end{aligned}$$

Conjecture. *Then* $\Psi(\mathrm{x}, \mathrm{y}, \mathrm{z}, b)$ *is the hypermap series where* b *marks an invariant of hypermaps*

Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ and a is an indeterminate. Let

$$\langle \cdot \rangle_{\mathbb{R}}^{a} : f \longmapsto \frac{\displaystyle\int_{\mathbb{R}^N} f(\lambda)\, |V(\lambda)|^{2a} e^{-\frac{a}{2} p_2(\lambda)} d\lambda}{\displaystyle\int_{\mathbb{R}^N} |V(\lambda)|^{2a} e^{-\frac{a}{2} p_2(\lambda)} d\lambda}$$

Then

$$[p_2^m] J_\theta(\lambda, a^{-1}) = \frac{\langle J_\theta(\lambda, a^{-1}) \rangle_N^a}{J_\theta(\mathbf{1}_N, a^{-1})}$$

Theorem. *Recall that b marks a conjectural invariant of maps. Then*

$$\begin{aligned} M(1, \mathrm{x}, N, z; b) &= \frac{2}{1+b} z \frac{\partial}{\partial z} \log \left\langle e^{(1+b) \sum_{k \geq 1} \frac{1}{k} z^{k/2} x_k p_k(\lambda)} \right\rangle_N^{1+b} \\ &\in \mathbb{Q}[\mathrm{x}, N, b][[z]] \end{aligned}$$

$M(1, \mathrm{x}, y, z; b) \in \mathbb{Q}[\mathrm{x}, y, b]\,[[z]]$ *is got by replacing N by y to mark faces*

II.4 - Example from the moduli space of curves

- Use Strebel quadratic derivative to obtain a cell decomposition of the moduli space of curves
- This gives its virtual Euler characteristic as a weighted sum of monopoles (maps with 1 vertex)
- The integration is carried out by the Sel'berg integration theorems

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Conjecture. *The conjectural map invariant marked by b has a geometric interpretation on the moduli space of curves*

Part III

Faber's Top Intersection Number Conjecture

Using this theorem we have:

Theorem (GJV)

$$K_\Theta^g = (-1)^k \langle \tau_{a_1}\tau_{a_2}\cdots\lambda_k\rangle_g \qquad \text{(Witten symbol)}$$

where

$$\langle \tau_{a_1}\tau_{a_2}\cdots\lambda_k\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1}\cdots\psi_m^{a_m}\lambda_k$$

$(\Theta = (a_1,\dots,a_m))$. $\square$

Corollary (GJV) Polynomiality

$$\langle \tau_{a_1}\tau_{a_2}\cdots\lambda_k\rangle_g = (-1)^k \left[\alpha_1^{a_1}\cdots\alpha_m^{a_m}\right] P_m^g(\alpha_1,\dots,\alpha_m)$$

where

P_m^g is a polynomial in $\alpha_1,\dots,\alpha_m$ with terms of degree between $2g-3+m$ and $3g-3+m = \dim \overline{\mathcal{M}}_{g,m}$. $\square$

We shall use this result in the Double Hurwitz Problem, although indirectly.

The ELSV theorem and its significance

Theorem (Ekedahl, Lando, Shapiro and Vainstein)

$$H_\alpha^g = C(g,\alpha) \int_{\overline{\mathcal{M}}_{g,n}} \frac{1-\lambda_1+\lambda_2-\cdots+(-1)^g\lambda_g}{\prod_{i=1}^{m}(1-\alpha_i\psi_i)}$$

where

$$C(g,\alpha) = r! \prod_{i=1}^{m} \frac{\alpha_i^{\alpha_i}}{\alpha_i!}$$

and

- $\overline{\mathcal{M}}_{g,m}$: Deligne-Mumford compactification of the moduli space of genus g curves with m marked points
- ψ_i : a certain codimension 1 class $(= c_1(\mathbb{L}_i))$ where
  - c_j is the j-th Chern class
  - $\mathbb{L}_i$ is a natural line bundle (cotangent space to the i-th marked point)
- λ_g : a certain codimension k class (Chow (cohomology) class) $(= c_k(\mathbb{E}))$ where
  - $\mathbb{E}$ is a natural rank g vector bundle. ☐

Note: Only the general form of this result will be needed in this talk.

III.1 - The conjecture

- Let $\overline{\mathcal{M}^{\mathrm{rt}}_{g,n}}$ be the compactification of the moduli space of genus g smooth curves (curves with "rational tails") with n marked points
- The intersection number for $\overline{\mathcal{M}^{\mathrm{rt}}_{g,n}}$ is denoted by

$$\langle \tau_{a_1} \cdots \tau_{a_n} \lambda_k \rangle_g^{\mathrm{rt}} \equiv \langle \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_k \rangle_g^{\mathrm{rt}}$$

where $a_1 + \cdots + a_n - k = g - 2 + n$ and ψ_i is a 1-dimensional Chow class and λ_k is a k-dimensional Chow class

- Top intersection numbers correspond to $k = 0$

Conjecture (Faber). *Let* $g \geq 3,\ a_1, \ldots, a_n \geq 1,\ a_1 + \cdots + a_n = g + n - 2.$ *Then*

$$\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g^{\mathrm{rt}} = \frac{(g-3+n)!(2g-1)!!}{(g-1)! \prod_{i=1}^{n} (2a_i - 1)!!}$$

where $k!! = 1 \cdot 3 \cdot 5 \cdots k$ *for* k *odd*

Approach I

- Getzler and Pandharipande proved that Faber's Top Intersection Number Conjecture is a consequence of the Virasoro Conjecture for $\mathbb{P}^2$
- Givental has outlined a proof of the Virasoro Conjecture for $\mathbb{P}^m$
- Thus a proof of Faber's Top Intersection Number Conjecture seems to be almost complete for (all g and all n)
- The proof is lengthy, dense and indirect

Approach II

- We (Goulden, DMRJ and Vakil) propose an alternative approach that we believe to be more direct
- The proof holds for all g (genera)
- The approach is through localisation theory from the work of Atiyah and Bott

III.2 - Outline of the proposed proof

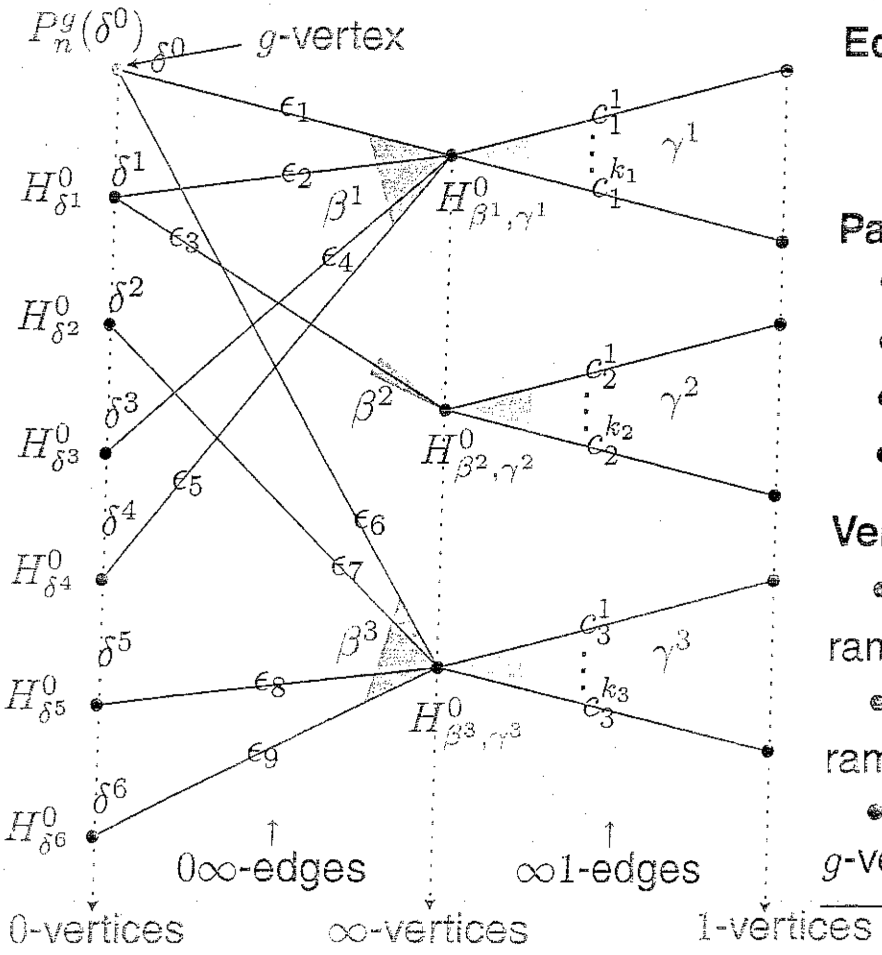
- We use localisation theory to express the Conjecture in terms of localisation trees, a tree weighted by Hurwitz Numbers, and the top potential
  - A Hurwitz Number counts ramified covers of the sphere by a curve of genus g with prescribed ramification over ∞ (and 0 in the case of the Double Hurwitz Number) and all other branch points elementary
  - A ramified cover can be encoded as a transitive ordered factorisation of a permutation into transpositions
- The top potential

$$P_n^g(\alpha) = \sum_{\substack{a_1,\dots,a_n \geq 0, \\ n-\delta_{g,1} \leq a_1+\cdots+a_n \leq g+n-2}} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g^{\mathrm{rt}} \alpha_1^{a_1} \cdots \alpha_n^{a_n}$$

satisfies a topological recursion

- We re-express Faber's Top Intersection Number Conjecture in terms of a formal differential system with underlying Lagrangian structures

III.3 - A localisation tree



Edge weights: integers > 0

- ϵ_i : on 0∞-edges
- c_i^j : on ∞1-edges

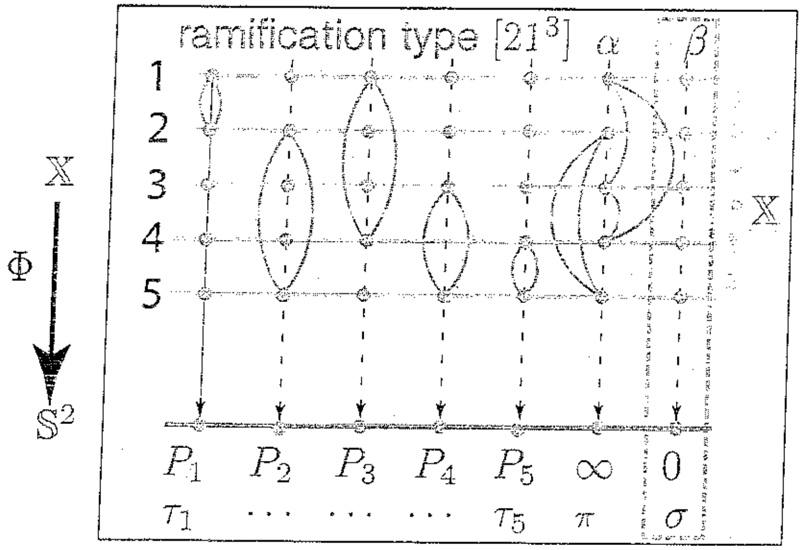
Partitions

- $\beta^1 = (\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5)$ etc
- $\gamma_i = (c_i^1, \ldots, c_i^{k_i})$ etc
- $\delta^0 = (\epsilon_1, \epsilon_6)$ etc
- Balance: $|\beta^i| = |\gamma^i|$

Vertex weights:

- $H^0_{\beta^i,\gamma^i}$: Hurwitz Number; ramification β at 0 & γ at ∞
- $H^0_{\delta^i}$ Hurwitz Number; ramification δ^i at ∞
- $P^g_n(\delta^0)$: Top potential at g-vertex

A ramified covers of the sphere



- $\mathbb{X}$: d-sheeted source curve
- $P_1, \ldots, P_r$: simple branch points
- τ_i, π, σ sheet transitions
- $\langle \tau_1, \ldots, \tau_r, \pi \rangle$ acts transitively on sheet labels $\{1, \ldots, d\}$ so $\mathbb{X}$ is connected
- $\tau_1 \cdots \tau_r \pi \sigma = \iota$: path from base point is contractible
- g: genus of $\mathbb{X}$

- In the example: Ignore objects in dotted box
  - $d = 5$ sheets; $r = 5$ simple branch points
  - $\tau_1 = (1,2), \tau_2 = (2,5), \tau_3 = (1,4), \tau_4 = (3,5), \tau_5 = (4,5)$
  - $\pi = (1,3,4)(2,5)$
  - ∞: ramification type $[3,2]$

Theorem. *The genus 0 Double Hurwitz Series H^0 defined by*

$$H^0(z, u : \mathbf{p}, \mathbf{q}) = \sum_{\alpha, \beta \in \mathcal{P}} \frac{H^0_{\alpha,\beta}}{r^0_{\alpha,\beta}! |\operatorname{Aut} \alpha| \, |\operatorname{Aut} \beta|} p_\alpha q_\beta z^{|\beta|} u^{l(\beta)}$$

satisfies the Join-Cut Equation

$$\sum_{i \geq 1} p_i \frac{\partial H^0}{\partial p_i} + u \frac{\partial H^0}{\partial u} - 2H^0 = \sum_{i,j \geq 1} \left(\frac{ij}{2} p_{i+j} \frac{\partial H^0}{\partial p_i} \frac{\partial H^0}{\partial p_j} + \frac{i+j}{2} p_i p_j \frac{\partial H^0}{\partial p_{i+j}} \right)$$

Definition. *The weighted potential Ψ^g is*

$$\begin{aligned}\Psi^g(z, u; \mathbf{p}) \;&=\; \sum_{n \geq 1} \sum_{\mathrm{t} \in \mathcal{T}_{g,n}} \frac{P^g_n(\delta^0)}{M! | \operatorname{Aut} \alpha|} \prod_{m \in \delta^0} \frac{m^m}{m!} \prod_i \varepsilon_i \prod_j{}' \frac{H^0_{\delta^j, (1^{|\delta^j|})}}{r^j_0! |\delta^j|!} \\ &\quad \cdot \prod_k \frac{H^0_{\beta^k, \gamma^k}}{r^k_\infty!} p_{\gamma^k} u^{l(\beta^k)-2} z^{|\gamma^k|}\end{aligned}$$

where the sum is over all genus g localisation trees

- The terms on the RHS correspond to :
- g-vertex.
- ∞-vertices,
- non-root 0-vertices,
- g-vertex,
- 0∞-edges

III.4 - The top intersection numbers $\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g^{\mathrm{rt}}$

Theorem. • *Let* $\xi^{(i)}(z,u;\mathbf{p}) = \sum_{j\geq 1} \frac{j^{j+i}}{j!} f_j(z,u;\mathbf{p})$ where f_j is given by

$$\begin{cases} f_j &= u^{-2} \left(j \dfrac{\partial}{\partial q_j} H^0(z,u;\mathbf{p};\mathbf{q}) \right) \Big|_{q_i = g_i, i \geq 1}, j \geq 1, \\ g_j &= \left(j \dfrac{\partial}{\partial q_j} H^0(1,1;\mathbf{e}_1;\mathbf{q}) \right) \Big|_{q_i = f_i, i \geq 1}, j \geq 1 \end{cases}$$

Then $\Psi^g(z,u;\mathbf{p}) = \sum_{n\geq 1} \frac{1}{n!} \sum_{a_1,\ldots,a_n \geq 0} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g^{\mathrm{rt}} \prod_{j=1}^n \xi^{(a_j)}(z,u;\mathbf{p})$

• *Let* F^g *be the solution of the partial differential equation*

$$\Delta F^g = \sum_{i,j\geq 1} \left(p_{i+j} \left(p_i^\star \widehat{H}^0 \right) p_j^\star + \frac{p_i p_j}{2} p_{i+j}^\star \right) F^g + \sum_{i\geq 1} i^{2g} p_i p_i^\star \widehat{H}^0$$

where $\Delta = z\frac{\partial}{\partial z} - 1 + \sum_{i\geq 1} p_i \frac{\partial}{\partial p_i}$ and $p_k^\star = k\partial/\partial p_k$ (adjoint action)

Then

$$c_g F^g(z;\mathbf{p}) = [u^{2g-1}]\Psi^g \left(z(1-u)^{-1}, -u; u(1-u)^{-1}\mathbf{p} \right)$$

III.5 - Reduction to a polynomial identity

- For a specific number of parts, this theorem reduces the proof of Faber's Top Intersection Number Conjecture to checking that two polynomials are identical

- The proof
  - is direct
  - holds for all genera g
  - holds for $n = 1, 2, 3$ at the moment
- Comments
  - We know the Double Hurwitz Numbers $H^0_{\alpha,\beta}$ for all α with $l(\alpha) \leq 5$ and for all β
  - We believe that we understand how to use the Join-Cut Equation for H^0 to avoid determining $H_{\alpha,\beta}$ explicitly.
  - This part of the work is in progress

End